1. It is easily verified that \( \langle e_m, e_n \rangle = \int_0^\pi (2/\pi)^{1/2} \sin(mx)(2/\pi)^{1/2} \sin(nx) = \delta_{mn} \). Hence the \( \{e_n\} \) form an orthonormal set. We may write 1 = \( \sum a_n e_n \) where \( a_n = \langle e_n, 1 \rangle = \int_0^\pi e_n = \sqrt{\frac{2}{\pi}} \left( \frac{2}{n} \right) \) for \( n \) odd and 0 for \( n \) even. Hence the Fourier series for \( f(x) = 1 \) is given by \( \sum \frac{4}{\pi(2n+1)} \sin((2n+1)x) \).

2. Evaluating the series above at \( x = \pi/2 \) we obtain \( 1 = \sum (-1)^n \frac{4}{\pi(2n+1)} \). Multiplying by \( \frac{\pi}{4} \) we get the desired identity.

Bonuses: Similarly evaluating using the function \( f(x) = x(\pi - x) \) gives \( 1 - \frac{1}{3^3} + \frac{1}{5^3} - \cdots = \frac{\pi^3}{32} \).

3. No. Note that the function \( f(x) = 1 \) is an element of \( L^2[0, \pi] \), but is orthogonal to all the \( e_n \).

4. (i) We may choose an orthonormal basis for \( S \). Note that this extends to an orthonormal basis for \( X \) (by first extending to a basis for \( X \), then orthonormalizing by Gram-Schmidt). Let \( e_1, e_2, \ldots \) be the chosen basis for \( S \), and \( f_1, f_2, \ldots \) be the elements of the orthonormal basis for \( X \) that are not in \( S \). Define \( T(x) = \sum \langle x, e_n \rangle e_n \). This sum converges by Bessel’s inequality, so \( T(x) \) is a well-defined element of \( S \) (since \( S \) is a closed subspace). For \( s \in S \) we have \( \langle T(x) - x, s \rangle = \langle \sum \langle x, f_n \rangle f_n, s \rangle = 0 \). To show uniqueness suppose \( T \) and \( T' \) both satisfy the conditions of the projection map. Then for any \( x \), \( 0 = \langle T(x) - x, s \rangle - \langle T'(x) - x, s \rangle = \langle T(x) - T'(x), s \rangle = 0 \) \( \forall s \in S \). Since \( T(x) - T'(x) \in S \) this implies \( T(x) = T'(x) \) \( \forall x \).

(ii) Linearity follows from bilinearity of the inner product, and \( \|T(x)\| \leq \|x\| \) follows from Bessel’s inequality.

(iii) Note that for any \( s \in S \), \( \|x - s\| = \|x - T(x)\| + \|T(x) - s\| \). This shows that \( T(x) \) is the unique point in \( S \) that is closest to \( x \).

5. (i) To see that \( X \) is closed, suppose \( f_n \to f \) in \( L^1 \), where \( f_n \in X \). Then \( f_n \to f \) in measure so a subsequence converges pointwise. This implies \( f \) can only take the values 0 and 1, so \( f \in X \) as well.
(ii) Let $S$ be the set of $\chi_E \in X$ that satisfy the given property. We must show $S$ contains a residual set. First note that if $I$ is a fixed interval in $[0, 1]$, the set $S_I = \{\chi_E | m(E \cap I), m(\tilde{E} \cap I) > 0\}$ is open and dense. It is open because if $\epsilon = \min\{m(E \cap I), m(\tilde{E} \cap I)\}$, then $B(\chi_E, \epsilon) \subset S_I$. It is dense because for any $\epsilon > 0$, if we let $E_\epsilon$ be a set of measure less than $\epsilon$ that belongs to $S_I$, then for any measurable $E$, $\chi_{E \cup E} \in B(\chi_E, \epsilon) \cap S_I$. Since $S = \cap_{I \in I} S_I$ where $I$ is the (countable) set of intervals in $[0, 1]$ with rational endpoints, by the Baire Category theorem $S$ is a residual set.

6. (i) To show $F_n$ is closed suppose $f_k \in F_n$ such that $f_k \to f$ in $C[0, 1]$ i.e. $f_k \to f$ uniformly. Let the $x_k$ be the points corresponding to the $f_k$. Since $\{x_k\}$ is bounded a subsequence must converge to a point $x_0$; rename $f_k$ to be this subsequence. We claim that $f$ satisfies $|f(x) - f(x_0)| \leq n|x - x_0|$ (*). This is true because for all $k$, $|f_k(x) - f_k(x_k)| \leq n|x - x_k|$ so taking limit, we get (*) (the limit of the LHS is $|f(x) - f(x_0)|$ because $f_n \to f$ uniformly). So $f \in F_n$, which shows that $F_n$ is closed. $F_n$ is nowhere dense because for any $f \in F_n$, and $\epsilon < 0$, we may take a continuous zigzag function $g$ with $f(x) - \epsilon < g(x) < f(x) + \epsilon$ satisfying $g'(x) > n$ wherever it exists; then note that $g \in B(f, \epsilon)$ but $g \notin F_n$. Thus $F_n$ is nowhere dense.

(ii) Suppose $f'(x_0)$ exists. It suffices to show $\sup_{x \in [0, 1]} \frac{|f(x) - f(x_0)|}{|x - x_0|} < \infty$ (*). Since $f$ is differentiable at $x_0$ there exists an open $U$ around $x_0$ for which $\sup_{x \in U} \frac{|f(x) - f(x_0)|}{|x - x_0|} < \infty$. But outside of $U$, $|x - x_0|$ is bounded below and the numerator is bounded above so this implies (*).

(iii) We must show that the set of nowhere differentiable functions in $C[0, 1]$ contains residual set. But this is true since its complement is contained in $\cup_n F_n$ which is meagre by the Baire Category theorem.