1. It suffices to show that every open set $U \subset \mathbb{R}$ is $F_{\sigma}$, since complements of $F_{\sigma}$ sets are $G_{\delta}$. If $U$ is open, then $U$ is a countable union of open intervals of the form $(a, b)$ where $a, b < \infty$. Since $F_{\sigma}$ sets are the countable union of closed sets it suffices to show that each interval $(a, b)$ is $F_{\sigma}$. However this is true because $(a, b) = \bigcup_{n \in \mathbb{N}} [a + \frac{1}{n}, b - \frac{1}{n}]$.

2. Solution 1. Let $I_1, \ldots, I_n$ be the given intervals, and define $A \subset [0, 1]$ such that $\bigcup I_i \cap [0, 1] = [0, 1] \setminus A$. We claim that $A$ is a null set. Note that if this is true, $\sum \ell(I_i) \geq m(\bigcup I_i) = m([0, 1] \setminus A) = 1$, as desired. To show that $A$ is measure 0, we see that if $x \in A$, then there exists $I_i$ such that $I_i = (x, y)$ for some $y \in \mathbb{R}$. Suppose the contrary. Then for some $\epsilon > 0$ we have $(x, x + \epsilon) \subset A$, but $(x, x + \epsilon) \cap \mathbb{Q}$ is nonempty since $\mathbb{Q}$ is dense, which is a contradiction. Thus for all $x \in A$ there exists $I_i$ such that $I_i = (x, y)$. Since there are only finite number of $I_i$, $A$ must be a finite set, thus null. So we have proved our claim.

Solution 2 sketch. One can also do this problem by writing length as an integral of the characteristic function (in the Riemann sense), and noting that each Riemann sum must be 1 since $1 \cap [0, 1] \leq 1 \cup I_i$ and $\mathbb{Q} \cap [0, 1]$ is dense in $[0, 1]$. This is similar to how it was proved in class that the lebesgue measure of an interval is its length.

3. Suppose $\mathbb{Q}$ is $G_{\delta}$. Then it can be written as $\cap U_i$. Order the elements of $\mathbb{Q}$ as $\{p_1, p_2, \ldots\}$. Let $V_i = U_i \setminus \{p_1, \ldots, p_i\}$. Note $U_1 \setminus \{p_1\}$ is open, so choose a compact interval $K_1 \subset U_1$ (in $\mathbb{R}$ with the usual topology, every open set contains a compact subset). Construct compact intervals $K_i$ inductively as follows. Since $K_{i-1}$ is a compact interval, it must contain a rational point that is not $\{p_1, \ldots, p_i\}$, (by density of $\mathbb{Q}$). So $K_{i-1} \cap V_i$ is nonempty, and since $K_{i-1}$ is an interval and $V_i$ is open, it contains a compact interval; choose one and let this interval be $K_i$. Note that $\cap K_i \neq \emptyset$ since $K_i$ is a sequence of nested compact intervals, but cannot contain a rational number by construction. This is a contradiction, thus $\mathbb{Q}$ is not $G_{\delta}$.

Remark. This a special case of a more general and very powerful result in analysis called the Baire Category theorem, which states that every complete metric space satisfies the property that given a countable
collection of $U_n$ open and dense, $\cap U_n$ is dense. The proof is similar to above, just done in a more general setting.

4. Let $U_n = \{x|\exists \delta \text{ s.t. } \forall y, z \in B(x, \delta), |f(y) - f(z)| < \frac{1}{n}\}$. By construction $U_n$ is open, and it is easy to see that $\cap U_n$ is the set of continuity points.

Remark. Many people wrote that the set $\{x|\exists \delta \text{ s.t. } \forall y \in B(x, \delta), f(y) \in B(f(x), \epsilon)\}$ is open. This is not necessarily true. For example, take the function $f(x) = -1$ if $x < 0$ is rational, $f(x) = 1$ if $x < 0$ is irrational and $f(x) = 0$ if $x \geq 0$.

5. The sequence $f_n(x)$ converges iff it is Cauchy. Thus the set of convergence points is $\cap_{n \in \mathbb{N}} \cup_{N=1}^{\infty} \cap_{m,n \geq N} C_{m,n,N}$ where $C_{m,n,N} = \{x||f_n(x) - f_m(x)| \leq \frac{1}{n}\}$. By continuity of $|f_n(x) - f_m(x)|$ we have that $C_{m,n,N}$ is closed, thus $\cap_{m,n \geq N} C_{m,n,N}$ is closed and we have expressed the set of convergence points as an $F_{\sigma \delta}$ set.

Remark. A few people involved $\lim_{n \to \infty} f_n(x)$ in the construction, despite the fact that this limit may not exist. One cannot union over all possible limits because there are uncountably many.

6. Note that the boundary points correspond precisely to the points which do not have a unique ternary expansion. Thus $0.20202... = \frac{1}{4}$ is not a boundary point, and is in the Cantor set since it does not contain a 1 in its ternary expansion.

Remark. Some people wrote that $0 \in K$ and is not the boundary point of a complementary interval, but this interpretation trivializes the problem.

7. Define $K$ similar to the Cantor set, by removing intervals of length $\frac{1}{2^k}$ from each of the $2^{k-1}$ intervals at the $k$th stage. We observe that total measure of the intervals removed is $\sum_{k=1}^{\infty} \frac{2^{k-2}}{3^k} = \frac{1}{2}$, thus $m(K) = \frac{1}{2} > 0$. This also implies that at each stage, the interval removed will be shorter than the interval it is removed from i.e. the set is well-defined. However $K$ has empty interior; to see this, note that at the $k$th stage, $K^c$ is the union of disjoint open intervals of length less than $2^{-i}$ (which we can see is inductively true).

8. We can assume $m(A) < \infty$ (otherwise, choose a subset). Note that there exists compact $K$ and open $U$ such that $K \subset A \subset U$ and $2m(K) > m(U)$. There exists an interval $I$ such that $K + I \subset U$. We claim that $I \subset A - A$. To see this, suppose $x \in I$ and $x \notin A - A$. 
Then $A$ and $A + x$ are disjoint, so in particular $K$ and $K + x$ are disjoint. Hence $m(K \cup (K + x)) = m(K) + m(K + x) = 2m(K)$. However $K \cup (K + x) \subset U$ so $m(K \cup (K + x)) \leq m(U)$, which is a contradiction. Therefore $I \subset A - A$. 