2 Basic Properties of Fourier Series

Nearly fifty years had passed without any progress on the question of analytic representation of an arbitrary function, when an assertion of Fourier threw new light on the subject. Thus a new era began for the development of this part of Mathematics and this was heralded in a stunning way by major developments in mathematical Physics.

_ B. Riemann, 1854_

In this chapter, we begin our rigorous study of Fourier series. We set the stage by introducing the main objects in the subject, and then formulate some basic problems which we have already touched upon earlier.

Our first result disposes of the question of uniqueness: Are two functions with the same Fourier coefficients necessarily equal? Indeed, a simple argument shows that if both functions are continuous, then in fact they must agree.

Next, we take a closer look at the partial sums of a Fourier series. Using the formula for the Fourier coefficients (which involves an integration), we make the key observation that these sums can be written conveniently as integrals:

\[ \frac{1}{2\pi} \int D_N(x - y)f(y) \, dy, \]

where \( \{D_N\} \) is a family of functions called the Dirichlet kernels. The above expression is the convolution of \( f \) with the function \( D_N \). Convolutions will play a critical role in our analysis. In general, given a family of functions \( \{K_n\} \), we are led to investigate the limiting properties as \( n \) tends to infinity of the convolutions

\[ \frac{1}{2\pi} \int K_n(x - y)f(y) \, dy. \]

We find that if the family \( \{K_n\} \) satisfies the three important properties of “good kernels,” then the convolutions above tend to \( f(x) \) as \( n \to \infty \) (at least when \( f \) is continuous). In this sense, the family \( \{K_n\} \) is an
“approximation to the identity.” Unfortunately, the Dirichlet kernels $D_N$ do not belong to the category of good kernels, which indicates that the question of convergence of Fourier series is subtle.

Instead of pursuing at this stage the problem of convergence, we consider various other methods of summing the Fourier series of a function. The first method, which involves averages of partial sums, leads to convolutions with good kernels, and yields an important theorem of Fejér. From this, we deduce the fact that a continuous function on the circle can be approximated uniformly by trigonometric polynomials. Second, we may also sum the Fourier series in the sense of Abel and again encounter a family of good kernels. In this case, the results about convolutions and good kernels lead to a solution of the Dirichlet problem for the steady-state heat equation in the disc, considered at the end of the previous chapter.

1 Examples and formulation of the problem

We commence with a brief description of the types of functions with which we shall be concerned. Since the Fourier coefficients of $f$ are defined by

$$a_n = \frac{1}{L} \int_0^L f(x)e^{-2\pi inx/L} \, dx, \quad \text{for } n \in \mathbb{Z},$$

where $f$ is complex-valued on $[0, L]$, it will be necessary to place some integrability conditions on $f$. We shall therefore assume for the remainder of this book that all functions are at least Riemann integrable.$^{1}$ Sometimes it will be illuminating to focus our attention on functions that are more “regular,” that is, functions that possess certain continuity or differentiability properties. Below, we list several classes of functions in increasing order of generality. We emphasize that we will not generally restrict our attention to real-valued functions, contrary to what the following pictures may suggest; we will almost always allow functions that take values in the complex numbers $\mathbb{C}$. Furthermore, we sometimes think of our functions as being defined on the circle rather than an interval. We elaborate upon this below.

$^{1}$Limiting ourselves to Riemann integrable functions is natural at this elementary stage of study of the subject. The more advanced notion of Lebesgue integrability will be taken up in Book III.
Everywhere continuous functions

These are the complex-valued functions $f$ which are continuous at every point of the segment $[0, L]$. A typical continuous function is sketched in Figure 1 (a). We shall note later that continuous functions on the circle satisfy the additional condition $f(0) = f(L)$.

Piecewise continuous functions

These are bounded functions on $[0, L]$ which have only finitely many discontinuities. An example of such a function with simple discontinuities is pictured in Figure 1 (b).

![Figure 1. Functions on [0, L]: continuous and piecewise continuous](image)

Riemann integrable functions

This is the most general class of functions we will be concerned with. Such functions are bounded, but may have infinitely many discontinuities. We recall the definition of integrability. A real-valued function $f$ defined on $[0, L]$ is Riemann integrable (which we abbreviate as integrable) if it is bounded, and if for every $\epsilon > 0$, there is a subdivision $0 = x_0 < x_1 < \cdots < x_{N-1} < x_N = L$ of the interval $[0, L]$, so that if $\mathcal{U}$

---

$^2$Starting in Book III, the term “integrable” will be used in the broader sense of Lebesgue theory.
and \( \mathcal{L} \) are, respectively, the upper and lower sums of \( f \) for this subdivision, namely

\[
\mathcal{U} = \sum_{j=1}^{N} \left[ \sup_{x_{j-1} \leq x \leq x_{j}} f(x) \right] (x_{j} - x_{j-1})
\]

and

\[
\mathcal{L} = \sum_{j=1}^{N} \left[ \inf_{x_{j-1} \leq x \leq x_{j}} f(x) \right] (x_{j} - x_{j-1})
\]

then we have \( \mathcal{U} - \mathcal{L} < \epsilon \). Finally, we say that a complex-valued function is integrable if its real and imaginary parts are integrable. It is worthwhile to remember at this point that the sum and product of two integrable functions are integrable.

A simple example of an integrable function on \([0, 1]\) with infinitely many discontinuities is given by

\[
f(x) = \begin{cases} 
1 & \text{if } 1/(n+1) < x \leq 1/n \text{ and } n \text{ is odd}, \\
0 & \text{if } 1/(n+1) < x \leq 1/n \text{ and } n \text{ is even}, \\
0 & \text{if } x = 0.
\end{cases}
\]

This example is illustrated in Figure 2. Note that \( f \) is discontinuous when \( x = 1/n \) and at \( x = 0 \).

![Figure 2. A Riemann integrable function](image)

More elaborate examples of integrable functions whose discontinuities are dense in the interval \([0, 1]\) are described in Problem 1. In general, while integrable functions may have infinitely many discontinuities, these
functions are actually characterized by the fact that, in a precise sense, their discontinuities are not too numerous: they are "negligible," that is, the set of points where an integrable function is discontinuous has "measure 0." The reader will find further details about Riemann integration in the appendix.

From now on, we shall always assume that our functions are integrable, even if we do not state this requirement explicitly.

**Functions on the circle**

There is a natural connection between $2\pi$-periodic functions on $\mathbb{R}$ like the exponentials $e^{in\theta}$, functions on an interval of length $2\pi$, and functions on the unit circle. This connection arises as follows.

A point on the unit circle takes the form $e^{i\theta}$, where $\theta$ is a real number that is unique up to integer multiples of $2\pi$. If $F$ is a function on the circle, then we may define for each real number $\theta$

$$f(\theta) = F(e^{i\theta}),$$

and observe that with this definition, the function $f$ is periodic on $\mathbb{R}$ of period $2\pi$, that is, $f(\theta + 2\pi) = f(\theta)$ for all $\theta$. The integrability, continuity and other smoothness properties of $F$ are determined by those of $f$. For instance, we say that $F$ is integrable on the circle if $f$ is integrable on every interval of length $2\pi$. Also, $F$ is continuous on the circle if $f$ is continuous on $\mathbb{R}$, which is the same as saying that $f$ is continuous on any interval of length $2\pi$. Moreover, $F$ is continuously differentiable if $f$ has a continuous derivative, and so forth.

Since $f$ has period $2\pi$, we may restrict it to any interval of length $2\pi$, say $[0, 2\pi]$ or $[-\pi, \pi]$, and still capture the initial function $F$ on the circle. We note that $f$ must take the same value at the end-points of the interval since they correspond to the same point on the circle. Conversely, any function on $[0, 2\pi]$ for which $f(0) = f(2\pi)$ can be extended to a periodic function on $\mathbb{R}$ which can then be identified as a function on the circle. In particular, a continuous function $f$ on the interval $[0, 2\pi]$ gives rise to a continuous function on the circle if and only if $f(0) = f(2\pi)$.

In conclusion, functions on $\mathbb{R}$ that $2\pi$-periodic, and functions on an interval of length $2\pi$ that take on the same value at its end-points, are two equivalent descriptions of the same mathematical objects, namely, functions on the circle.

In this connection, we mention an item of notational usage. When our functions are defined on an interval on the line, we often use $x$ as the independent variable; however, when we consider these as functions
on the circle, we usually replace the variable $x$ by $\theta$. As the reader will note, we are not strictly bound by this rule since this practice is mostly a matter of convenience.

1.1 Main definitions and some examples

We now begin our study of Fourier analysis with the precise definition of the Fourier series of a function. Here, it is important to pin down where our function is originally defined. If $f$ is an integrable function given on an interval $[a, b]$ of length $L$ (that is, $b - a = L$), then the $n^{th}$ Fourier coefficient of $f$ is defined by

$$\hat{f}(n) = \frac{1}{L} \int_{a}^{b} f(x)e^{-2\pi inx/L} \, dx, \quad n \in \mathbb{Z}.$$  

The Fourier series of $f$ is given formally$^3$ by

$$\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{2\pi inx/L}.$$  

We shall sometimes write $a_n$ for the Fourier coefficients of $f$, and use the notation

$$f(x) \sim \sum_{n=-\infty}^{\infty} a_n e^{2\pi inx/L}$$

to indicate that the series on the right-hand side is the Fourier series of $f$.

For instance, if $f$ is an integrable function on the interval $[-\pi, \pi]$, then the $n^{th}$ Fourier coefficient of $f$ is

$$\hat{f}(n) = a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)e^{-in\theta} \, d\theta, \quad n \in \mathbb{Z},$$

and the Fourier series of $f$ is

$$f(\theta) \sim \sum_{n=-\infty}^{\infty} a_n e^{in\theta}.$$  

Here we use $\theta$ as a variable since we think of it as an angle ranging from $-\pi$ to $\pi$.

---

$^3$At this point, we do not say anything about the convergence of the series.
Also, if \( f \) is defined on \([0, 2\pi]\), then the formulas are the same as above, except that we integrate from 0 to \(2\pi\) in the definition of the Fourier coefficients.

We may also consider the Fourier coefficients and Fourier series for a function defined on the circle. By our previous discussion, we may think of a function on the circle as a function \( f \) on \( \mathbb{R} \) which is \(2\pi\)-periodic. We may restrict the function \( f \) to any interval of length \(2\pi\), for instance \([0, 2\pi]\) or \([-\pi, \pi]\), and compute its Fourier coefficients. Fortunately, \( f \) is periodic and Exercise 1 shows that the resulting integrals are independent of the chosen interval. Thus the Fourier coefficients of a function on the circle are well defined.

Finally, we shall sometimes consider a function \( g \) given on \([0, 1]\). Then

\[
\hat{g}(n) = a_n = \int_0^1 g(x)e^{-2\pi inx} \, dx \quad \text{and} \quad g(x) \sim \sum_{n=-\infty}^{\infty} a_n e^{2\pi inx}.
\]

Here we use \( x \) for a variable ranging from 0 to 1.

Of course, if \( f \) is initially given on \([0, 2\pi]\), then \( g(x) = f(2\pi x) \) is defined on \([0, 1]\) and a change of variables shows that the \( n^{\text{th}} \) Fourier coefficient of \( f \) equals the \( n^{\text{th}} \) Fourier coefficient of \( g \).

Fourier series are part of a larger family called the trigonometric series which, by definition, are expressions of the form \( \sum_{n=-\infty}^{\infty} c_n e^{2\pi inx/L} \) where \( c_n \in \mathbb{C} \). If a trigonometric series involves only finitely many non-zero terms, that is, \( c_n = 0 \) for all large \(|n|\), it is called a trigonometric polynomial; its degree is the largest value of \(|n|\) for which \( c_n \neq 0 \).

The \( N^{\text{th}} \) partial sum of the Fourier series of \( f \), for \( N \) a positive integer, is a particular example of a trigonometric polynomial. It is given by

\[
S_N(f)(x) = \sum_{n=-N}^{N} \hat{f}(n)e^{2\pi inx/L}.
\]

Note that by definition, the above sum is symmetric since \( n \) ranges from \(-N\) to \(N\), a choice that is natural because of the resulting decomposition of the Fourier series as sine and cosine series. As a consequence, the convergence of Fourier series will be understood (in this book) as the “limit” as \( N \) tends to infinity of these symmetric sums.

In fact, using the partial sums of the Fourier series, we can reformulate the basic question raised in Chapter 1 as follows:

**Problem:** In what sense does \( S_N(f) \) converge to \( f \) as \( N \to \infty \)?
Before proceeding further with this question, we turn to some simple examples of Fourier series.

**Example 1.** Let \( f(\theta) = \theta \) for \(-\pi \leq \theta \leq \pi\). The calculation of the Fourier coefficients requires a simple integration by parts. First, if \( n \neq 0 \), then

\[
\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \theta e^{-in\theta} \, d\theta
\]

\[
= \frac{1}{2\pi} \left[ \frac{\theta}{in} e^{-in\theta} \right]_{-\pi}^{\pi} + \frac{1}{2in} \int_{-\pi}^{\pi} e^{-in\theta} \, d\theta
\]

\[
= \frac{(-1)^{n+1}}{in},
\]

and if \( n = 0 \) we clearly have

\[
\hat{f}(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \theta \, d\theta = 0.
\]

Hence, the Fourier series of \( f \) is given by

\[
f(\theta) \sim \sum_{n \neq 0} \frac{(-1)^{n+1}}{in} e^{in\theta} = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin n\theta}{n}.
\]

The first sum is over all non-zero integers, and the second is obtained by an application of Euler’s identities. It is possible to prove by elementary means that the above series converges for every \( \theta \), but it is not obvious that it converges to \( f(\theta) \). This will be proved later (Exercises 8 and 9 deal with a similar situation).

**Example 2.** Define \( f(\theta) = (\pi - \theta)^2/4 \) for \( 0 \leq \theta \leq 2\pi \). Then successive integration by parts similar to that performed in the previous example yield

\[
f(\theta) \sim \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{\cos n\theta}{n^2}.
\]

**Example 3.** The Fourier series of the function

\[
f(\theta) = \frac{\pi}{\sin \pi \alpha} e^{i(\pi - \theta)\alpha}
\]

on \([0, 2\pi]\) is

\[
f(\theta) \sim \sum_{n=-\infty}^{\infty} \frac{e^{in\theta}}{n + \alpha},
\]
whenever $\alpha$ is not an integer.

**Example 4.** The trigonometric polynomial defined for $x \in [-\pi, \pi]$ by

$$D_N(x) = \sum_{n=-N}^{N} e^{inx}$$

is called the $N^{th}$ **Dirichlet kernel** and is of fundamental importance in the theory (as we shall see later). Notice that its Fourier coefficients $a_n$ have the property that $a_n = 1$ if $|n| \leq N$ and $a_n = 0$ otherwise. A closed form formula for the Dirichlet kernel is

$$D_N(x) = \frac{\sin((N + \frac{1}{2})x)}{\sin(x/2)}.$$

This can be seen by summing the geometric progressions

$$\sum_{n=0}^{N} \omega^n \quad \text{and} \quad \sum_{n=-N}^{-1} \omega^n$$

with $\omega = e^{ix}$. These sums are, respectively, equal to

$$\frac{1 - \omega^{N+1}}{1 - \omega} \quad \text{and} \quad \frac{\omega^{-N} - 1}{1 - \omega}.$$

Their sum is then

$$\frac{\omega^{-N} - \omega^{N+1}}{1 - \omega} = \frac{\omega^{-N-1/2} - \omega^{N+1/2}}{\omega^{-1/2} - \omega^{1/2}} = \frac{\sin((N + \frac{1}{2})x)}{\sin(x/2)},$$

giving the desired result.

**Example 5.** The function $P_r(\theta)$, called the **Poisson kernel**, is defined for $\theta \in [-\pi, \pi]$ and $0 \leq r < 1$ by the absolutely and uniformly convergent series

$$P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}.$$ 

This function arose implicitly in the solution of the steady-state heat equation on the unit disc discussed in Chapter 1. Note that in calculating the Fourier coefficients of $P_r(\theta)$ we can interchange the order of integration and summation since the sum converges uniformly in $\theta$ for
each fixed $r$, and obtain that the $n^{th}$ Fourier coefficient equals $r^{|n|}$. One can also sum the series for $P_r(\theta)$ and see that

$$P_r(\theta) = \frac{1 - r^2}{1 - 2r \cos \theta + r^2}.$$

In fact,

$$P_r(\theta) = \sum_{n=0}^{\infty} \omega^n + \sum_{n=1}^{\infty} \overline{\omega}^n \quad \text{with} \quad \omega = re^{i\theta},$$

where both series converge absolutely. The first sum (an infinite geometric progression) equals $1/(1 - \omega)$, and likewise, the second is $\overline{\omega}/(1 - \overline{\omega})$. Together, they combine to give

$$\frac{1 - \overline{\omega} + (1 - \omega)\overline{\omega}}{(1 - \omega)(1 - \overline{\omega})} = \frac{1 - |\omega|^2}{|1 - \omega|^2} = \frac{1 - r^2}{1 - 2r \cos \theta + r^2},$$

as claimed. The Poisson kernel will reappear later in the context of Abel summability of the Fourier series of a function.

Let us return to the problem formulated earlier. The definition of the Fourier series of $f$ is purely formal, and it is not obvious whether it converges to $f$. In fact, the solution of this problem can be very hard, or relatively easy, depending on the sense in which we expect the series to converge, or on what additional restrictions we place on $f$.

Let us be more precise. Suppose, for the sake of this discussion, that the function $f$ (which is always assumed to be Riemann integrable) is defined on $[-\pi, \pi]$. The first question one might ask is whether the partial sums of the Fourier series of $f$ converge to $f$ pointwise. That is, do we have

$$\lim_{N \to \infty} S_N(f)(\theta) = f(\theta) \quad \text{for every } \theta? \tag{1}$$

We see quite easily that in general we cannot expect this result to be true at every $\theta$, since we can always change an integrable function at one point without changing its Fourier coefficients. As a result, we might ask the same question assuming that $f$ is continuous and periodic. For a long time it was believed that under these additional assumptions the answer would be “yes.” It was a surprise when Du Bois-Reymond showed that there exists a continuous function whose Fourier series diverges at a point. We will give such an example in the next chapter. Despite this negative result, we might ask what happens if we add more smoothness conditions on $f$: for example, we might assume that $f$ is continuously
differentiable, or twice continuously differentiable. We will see that then the Fourier series of \( f \) converges to \( f \) uniformly.

We will also interpret the limit (1) by showing that the Fourier series sums, in the sense of Cesàro or Abel, to the function \( f \) at all of its points of continuity. This approach involves appropriate averages of the partial sums of the Fourier series of \( f \).

Finally, we can also define the limit (1) in the mean square sense. In the next chapter, we will show that if \( f \) is merely integrable, then

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} |S_N(f)(\theta) - f(\theta)|^2 \, d\theta \to 0 \quad \text{as } N \to \infty.
\]

It is of interest to know that the problem of pointwise convergence of Fourier series was settled in 1966 by L. Carleson, who showed, among other things, that if \( f \) is integrable in our sense,\(^4\) then the Fourier series of \( f \) converges to \( f \) except possibly on a set of “measure 0.” The proof of this theorem is difficult and beyond the scope of this book.

2 Uniqueness of Fourier series

If we were to assume that the Fourier series of functions \( f \) converge to \( f \) in an appropriate sense, then we could infer that a function is uniquely determined by its Fourier coefficients. This would lead to the following statement: if \( f \) and \( g \) have the same Fourier coefficients, then \( f \) and \( g \) are necessarily equal. By taking the difference \( f - g \), this proposition can be reformulated as: if \( \hat{f}(n) = 0 \) for all \( n \in \mathbb{Z} \), then \( f = 0 \). As stated, this assertion cannot be correct without reservation, since calculating Fourier coefficients requires integration, and we see that, for example, any two functions which differ at finitely many points have the same Fourier series. However, we do have the following positive result.

**Theorem 2.1** Suppose that \( f \) is an integrable function on the circle with \( \hat{f}(n) = 0 \) for all \( n \in \mathbb{Z} \). Then \( f(\theta_0) = 0 \) whenever \( f \) is continuous at the point \( \theta_0 \).

Thus, in terms of what we know about the set of discontinuities of integrable functions,\(^5\) we can conclude that \( f \) vanishes for “most” values of \( \theta \).

**Proof.** We suppose first that \( f \) is real-valued, and argue by contradiction. Assume, without loss of generality, that \( f \) is defined on

\(^4\)Carleson’s proof actually holds for the wider class of functions which are square integrable in the Lebesgue sense.

\(^5\)See the appendix.
$[-\pi, \pi]$, that $\theta_0 = 0$, and $f(0) > 0$. The idea now is to construct a family of trigonometric polynomials $\{p_k\}$ that “peak” at 0, and so that $\int p_k(\theta)f(\theta)\,d\theta \to \infty$ as $k \to \infty$. This will be our desired contradiction since these integrals are equal to zero by assumption.

Since $f$ is continuous at 0, we can choose $0 < \delta \leq \pi/2$, so that $f(\theta) > f(0)/2$ whenever $|\theta| < \delta$. Let

$$p(\theta) = \epsilon + \cos \theta,$$

where $\epsilon > 0$ is chosen so small that $|p(\theta)| < 1 - \epsilon/2$, whenever $\delta \leq |\theta| \leq \pi$. Then, choose a positive $\eta$ with $\eta < \delta$, so that $p(\theta) \geq 1 + \epsilon/2$, for $|\theta| < \eta$. Finally, let

$$p_k(\theta) = [p(\theta)]^k,$$

and select $B$ so that $|f(\theta)| \leq B$ for all $\theta$. This is possible since $f$ is integrable, hence bounded. Figure 3 illustrates the family $\{p_k\}$. By

![Figure 3. The functions $p$, $p_6$, and $p_{15}$ when $\epsilon = 0.1$](image)

construction, each $p_k$ is a trigonometric polynomial, and since $\hat{f}(n) = 0$ for all $n$, we must have

$$\int_{-\pi}^{\pi} f(\theta)p_k(\theta)\,d\theta = 0 \quad \text{for all } k.$$

However, we have the estimate

$$\left| \int_{\delta \leq |\theta|} f(\theta)p_k(\theta)\,d\theta \right| \leq 2\pi B(1 - \epsilon/2)^k.$$
Also, our choice of $\delta$ guarantees that $p(\theta)$ and $f(\theta)$ are non-negative whenever $|\theta| < \delta$, thus

$$\int_{|\theta| < \delta} f(\theta)p_k(\theta) \, d\theta \geq 0.$$ 

Finally,

$$\int_{|\theta| < \eta} f(\theta)p_k(\theta) \, d\theta \geq 2\eta \frac{f(0)}{2} (1 + \epsilon/2)^k.$$ 

Therefore, $\int p_k(\theta)f(\theta) \, d\theta \to \infty$ as $k \to \infty$, and this concludes the proof when $f$ is real-valued. In general, write $f(\theta) = u(\theta) + iv(\theta)$, where $u$ and $v$ are real-valued. If we define $\bar{f}(\theta) = \overline{f(\theta)}$, then

$$u(\theta) = \frac{f(\theta) + \bar{f}(\theta)}{2} \quad \text{and} \quad v(\theta) = \frac{f(\theta) - \bar{f}(\theta)}{2i},$$

and since $\hat{f}(n) = \overline{\hat{f}(-n)}$, we conclude that the Fourier coefficients of $u$ and $v$ all vanish, hence $f = 0$ at its points of continuity. The idea of constructing a family of functions (trigonometric polynomials in this case) which peak at the origin, together with other nice properties, will play an important role in this book. Such families of functions will be taken up later in Section 4 in connection with the notion of convolution. For now, note that the above theorem implies the following.

**Corollary 2.2** If $f$ is continuous on the circle and $\hat{f}(n) = 0$ for all $n \in \mathbb{Z}$, then $f = 0$.

The next corollary shows that the problem (1) formulated earlier has a simple positive answer under the assumption that the series of Fourier coefficients converges absolutely.

**Corollary 2.3** Suppose that $f$ is a continuous function on the circle and that the Fourier series of $f$ is absolutely convergent, $\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty$. Then, the Fourier series converges uniformly to $f$, that is,

$$\lim_{N \to \infty} S_N(f)(\theta) = f(\theta) \quad \text{uniformly in} \ \theta.$$ 

**Proof.** Recall that if a sequence of continuous functions converges uniformly, then the limit is also continuous. Now observe that the assumption $\sum |\hat{f}(n)| < \infty$ implies that the partial sums of the Fourier
series of $f$ converge absolutely and uniformly, and therefore the function $g$ defined by

$$g(\theta) = \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{in\theta} = \lim_{N \to \infty} \sum_{n=-N}^{N} \hat{f}(n)e^{in\theta}$$

is continuous on the circle. Moreover, the Fourier coefficients of $g$ are precisely $\hat{f}(n)$ since we can interchange the infinite sum with the integral (a consequence of the uniform convergence of the series). Therefore, the previous corollary applied to the function $f - g$ yields $f = g$, as desired.

What conditions on $f$ would guarantee the absolute convergence of its Fourier series? As it turns out, the smoothness of $f$ is directly related to the decay of the Fourier coefficients, and in general, the smoother the function, the faster this decay. As a result, we can expect that relatively smooth functions equal their Fourier series. This is in fact the case, as we now show.

In order to state the result concisely we introduce the standard "$O$" notation, which we will use freely in the rest of this book. For example, the statement $\hat{f}(n) = O(1/|n|^2)$ as $|n| \to \infty$, means that the left-hand side is bounded by a constant multiple of the right-hand side; that is, there exists $C > 0$ with $|\hat{f}(n)| \leq C/|n|^2$ for all large $|n|$. More generally, $f(x) = O(g(x))$ as $x \to a$ means that for some constant $C$, $|f(x)| \leq C|g(x)|$ as $x$ approaches $a$. In particular, $f(x) = O(1)$ means that $f$ is bounded.

**Corollary 2.4** Suppose that $f$ is a twice continuously differentiable function on the circle. Then

$$\hat{f}(n) = O(1/|n|^2) \quad \text{as } |n| \to \infty,$$

so that the Fourier series of $f$ converges absolutely and uniformly to $f$. 
Proof. The estimate on the Fourier coefficients is proved by integrating by parts twice for \( n \neq 0 \). We obtain

\[
2\pi \hat{f}(n) = \int_0^{2\pi} f(\theta)e^{-in\theta} \, d\theta \\
= \left[ f(\theta) \cdot \frac{-e^{-in\theta}}{in} \right]_0^{2\pi} + \frac{1}{in} \int_0^{2\pi} f'(\theta)e^{-in\theta} \, d\theta \\
= \frac{1}{in} \int_0^{2\pi} f'(\theta)e^{-in\theta} \, d\theta \\
= \frac{1}{in} \left[ f'(\theta) \cdot \frac{-e^{-in\theta}}{in} \right]_0^{2\pi} + \frac{1}{(in)^2} \int_0^{2\pi} f''(\theta)e^{-in\theta} \, d\theta \\
= -\frac{1}{n^2} \int_0^{2\pi} f''(\theta)e^{-in\theta} \, d\theta.
\]

The quantities in brackets vanish since \( f \) and \( f' \) are periodic. Therefore

\[
2\pi |n|^2 |\hat{f}(n)| \leq \left| \int_0^{2\pi} f''(\theta)e^{-in\theta} \, d\theta \right| \leq \int_0^{2\pi} |f''(\theta)| \, d\theta \leq C,
\]

where the constant \( C \) is independent of \( n \). (We can take \( C = 2\pi B \) where \( B \) is a bound for \( f'' \).) Since \( \sum 1/n^2 \) converges, the proof of the corollary is complete.

Incidentally, we have also established the following important identity:

\[
\hat{f}'(n) = in \hat{f}(n), \quad \text{for all } n \in \mathbb{Z}.
\]

If \( n \neq 0 \) the proof is given above, and if \( n = 0 \) it is left as an exercise to the reader. So if \( f \) is differentiable and \( f \sim \sum a_n e^{in\theta} \), then \( f' \sim \sum a_n ine^{in\theta} \). Also, if \( f \) is twice continuously differentiable, then \( f'' \sim \sum a_n (in)^2 e^{in\theta} \), and so on. Further smoothness conditions on \( f \) imply even better decay of the Fourier coefficients (Exercise 10).

There are also stronger versions of Corollary 2.4. It can be shown, for example, that the Fourier series of \( f \) converges absolutely, assuming only that \( f \) has one continuous derivative. Even more generally, the Fourier series of \( f \) converges absolutely (and hence uniformly to \( f \)) if \( f \) satisfies a Hölder condition of order \( \alpha \), with \( \alpha > 1/2 \), that is,

\[
\sup_\theta |f(\theta + t) - f(\theta)| \leq A|t|^\alpha \quad \text{for all } t.
\]

For more on these matters, see the exercises at the end of Chapter 3.
At this point it is worthwhile to introduce a common notation: we say that \( f \) belongs to the class \( C^k \) if \( f \) is \( k \) times continuously differentiable. Belonging to the class \( C^k \) or satisfying a Hölder condition are two possible ways to describe the smoothness of a function.

3 Convolutions

The notion of convolution of two functions plays a fundamental role in Fourier analysis; it appears naturally in the context of Fourier series but also serves more generally in the analysis of functions in other settings.

Given two \( 2\pi \)-periodic integrable functions \( f \) and \( g \) on \( \mathbb{R} \), we define their convolution \( f \ast g \) on \( [-\pi, \pi] \) by

\[
(f \ast g)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y)g(x - y) \, dy.
\]

(2)

The above integral makes sense for each \( x \), since the product of two integrable functions is again integrable. Also, since the functions are periodic, we can change variables to see that

\[
(f \ast g)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x - y)g(y) \, dy.
\]

Loosely speaking, convolutions correspond to “weighted averages.” For instance, if \( g = 1 \) in (2), then \( f \ast g \) is constant and equal to \( \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) \, dy \), which we may interpret as the average value of \( f \) on the circle. Also, the convolution \( (f \ast g)(x) \) plays a role similar to, and in some sense replaces, the pointwise product \( f(x)g(x) \) of the two functions \( f \) and \( g \).

In the context of this chapter, our interest in convolutions originates from the fact that the partial sums of the Fourier series of \( f \) can be expressed as follows:

\[
S_N(f)(x) = \sum_{n=-N}^{N} \hat{f}(n)e^{inx}
\]

\[
= \sum_{n=-N}^{N} \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y)e^{-iny} \, dy \right) e^{inx}
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) \left( \sum_{n=-N}^{N} e^{in(x-y)} \right) \, dy
\]

\[
= (f \ast D_N)(x),
\]
where $D_N$ is the $N^{th}$ Dirichlet kernel (see Example 4) given by

$$D_N(x) = \sum_{n=-N}^{N} e^{inx}.$$ 

So we observe that the problem of understanding $S_N(f)$ reduces to the understanding of the convolution $f \ast D_N$.

We begin by gathering some of the main properties of convolutions.

**Proposition 3.1** Suppose that $f$, $g$, and $h$ are $2\pi$-periodic integrable functions. Then:

(i) $f \ast (g + h) = (f \ast g) + (f \ast h)$.

(ii) $(cf) \ast g = c(f \ast g) = f \ast (cg)$ for any $c \in \mathbb{C}$.

(iii) $f \ast g = g \ast f$.

(iv) $(f \ast g) \ast h = f \ast (g \ast h)$.

(v) $f \ast g$ is continuous.

(vi) $\hat{f \ast g(n)} = \hat{f}(n)\hat{g}(n)$.

The first four points describe the algebraic properties of convolutions: linearity, commutativity, and associativity. Property (v) exhibits an important principle: the convolution of $f \ast g$ is "more regular" than $f$ or $g$. Here, $f \ast g$ is continuous while $f$ and $g$ are merely (Riemann) integrable. Finally, (vi) is key in the study of Fourier series. In general, the Fourier coefficients of the product $fg$ are not the product of the Fourier coefficients of $f$ and $g$. However, (vi) says that this relation holds if we replace the product of the two functions $f$ and $g$ by their convolution $f \ast g$.

**Proof.** Properties (i) and (ii) follow at once from the linearity of the integral.

The other properties are easily deduced if we assume also that $f$ and $g$ are continuous. In this case, we may freely interchange the order of
integration. For instance, to establish (vi) we write
\[
\hat{f} \ast \hat{g}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (f \ast g)(x) e^{-inx} \, dx
\]
\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{2\pi} \left( \int_{-\pi}^{\pi} f(y) g(x - y) \, dy \right) e^{-inx} \, dx
\]
\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{-iny} \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} g(x - y) e^{-in(x-y)} \, dx \right) \, dy
\]
\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{-iny} \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} g(x) e^{-inx} \, dx \right) \, dy
\]
\[
= \hat{f}(n) \hat{g}(n).
\]

To prove (iii), one first notes that if \( F \) is continuous and \( 2\pi \)-periodic, then
\[
\int_{-\pi}^{\pi} F(y) \, dy = \int_{-\pi}^{\pi} F(x - y) \, dy \quad \text{for any } x \in \mathbb{R}.
\]
The verification of this identity consists of a change of variables \( y \mapsto -y \), followed by a translation \( y \mapsto y - x \). Then, one takes \( F(y) = f(y) g(x - y) \).

Also, (iv) follows by interchanging two integral signs, and an appropriate change of variables.

Finally, we show that if \( f \) and \( g \) are continuous, then \( f \ast g \) is continuous. First, we may write
\[
(f \ast g)(x_1) - (f \ast g)(x_2) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) [g(x_1 - y) - g(x_2 - y)] \, dy.
\]
Since \( g \) is continuous it must be uniformly continuous on any closed and bounded interval. But \( g \) is also periodic, so it must be uniformly continuous on all of \( \mathbb{R} \); given \( \epsilon > 0 \) there exists \( \delta > 0 \) so that \( |g(s) - g(t)| < \epsilon \) whenever \( |s - t| < \delta \). Then, \( |x_1 - x_2| < \delta \) implies \( |(x_1 - y) - (x_2 - y)| < \delta \) for any \( y \), hence
\[
|f \ast g)(x_1) - (f \ast g)(x_2)| \leq \frac{1}{2\pi} \left| \int_{-\pi}^{\pi} f(y) [g(x_1 - y) - g(x_2 - y)] \, dy \right|
\]
\[
\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(y)| |g(x_1 - y) - g(x_2 - y)| \, dy
\]
\[
\leq \frac{\epsilon}{2\pi} \int_{-\pi}^{\pi} |f(y)| \, dy
\]
\[
\leq \frac{\epsilon}{2\pi} 2\pi B,
\]
where $B$ is chosen so that $|f(x)| \leq B$ for all $x$. As a result, we conclude that $f \ast g$ is continuous, and the proposition is proved, at least when $f$ and $g$ are continuous.

In general, when $f$ and $g$ are merely integrable, we may use the results established so far (when $f$ and $g$ are continuous), together with the following approximation lemma, whose proof may be found in the appendix.

**Lemma 3.2** Suppose $f$ is integrable on the circle and bounded by $B$. Then there exists a sequence $\{f_k\}_{k=1}^{\infty}$ of continuous functions on the circle so that

$$\sup_{x \in [-\pi, \pi]} |f_k(x)| \leq B \quad \text{for all } k = 1, 2, \ldots,$$

and

$$\int_{-\pi}^{\pi} |f(x) - f_k(x)| \, dx \to 0 \quad \text{as } k \to \infty.$$

Using this result, we may complete the proof of the proposition as follows. Apply Lemma 3.2 to $f$ and $g$ to obtain sequences $\{f_k\}$ and $\{g_k\}$ of approximating continuous functions. Then

$$f \ast g - f_k \ast g_k = (f - f_k) \ast g + f_k \ast (g - g_k).$$

By the properties of the sequence $\{f_k\}$,

$$|(f - f_k) \ast g(x)| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x - y) - f_k(x - y)| \, |g(y)| \, dy$$

$$\leq \frac{1}{2\pi} \sup_{y} |g(y)| \int_{-\pi}^{\pi} |f(y) - f_k(y)| \, dy$$

$$\to 0 \quad \text{as } k \to \infty.$$

Hence $(f - f_k) \ast g \to 0$ uniformly in $x$. Similarly, $f_k \ast (g - g_k) \to 0$ uniformly, and therefore $f_k \ast g_k$ tends uniformly to $f \ast g$. Since each $f_k \ast g_k$ is continuous, it follows that $f \ast g$ is also continuous, and we have (v).

Next, we establish (vi). For each fixed integer $n$ we must have $\hat{f_k} \ast \hat{g_k}(n) \to \hat{f} \ast \hat{g}(n)$ as $k$ tends to infinity since $f_k \ast g_k$ converges uniformly to $f \ast g$. However, we found earlier that $\hat{f_k}(n) \hat{g_k}(n) = \hat{f_k} \ast \hat{g_k}(n)$ because both $f_k$ and $g_k$ are continuous. Hence

$$|\hat{f}(n) - \hat{f_k}(n)| = \frac{1}{2\pi} \left| \int_{-\pi}^{\pi} (f(x) - f_k(x)) e^{-inx} \, dx \right|$$

$$\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - f_k(x)| \, dx,$$
and as a result we find that \( \widehat{f}_k(n) \to \hat{f}(n) \) as \( k \) goes to infinity. Similarly 
\( g_k(n) \to \check{g}(n) \), and the desired property is established once we let \( k \) tend 
to infinity. Finally, properties (iii) and (iv) follow from the same kind of 
arguments.

4 Good kernels

In the proof of Theorem 2.1 we constructed a sequence of trigonometric 
polynomials \( \{p_k\} \) with the property that the functions \( p_k \) peaked at the 
origin. As a result, we could isolate the behavior of \( f \) at the origin. In 
this section, we return to such families of functions, but this time in a 
more general setting. First, we define the notion of good kernel, and 
discuss the characteristic properties of such functions. Then, by the use 
of convolutions, we show how these kernels can be used to recover a given 
function.

A family of kernels \( \{K_n(x)\}_{n=1}^{\infty} \) on the circle is said to be a family of 
good kernels if it satisfies the following properties:

(a) For all \( n \geq 1 \),
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(x) \, dx = 1.
\]

(b) There exists \( M > 0 \) such that for all \( n \geq 1 \),
\[
\int_{-\pi}^{\pi} |K_n(x)| \, dx \leq M.
\]

(c) For every \( \delta > 0 \),
\[
\int_{\delta \leq |x| \leq \pi} |K_n(x)| \, dx \to 0, \quad \text{as} \ n \to \infty.
\]

In practice we shall encounter families where \( K_n(x) \geq 0 \), in which 
case (b) is a consequence of (a). We may interpret the kernels \( K_n(x) \) 
as weight distributions on the circle: property (a) says that \( K_n \) assigns 
unit mass to the whole circle \([-\pi, \pi]\), and (c) that this mass concentrates 
near the origin as \( n \) becomes large.\(^6\) Figure 4 (a) illustrates the typical 
character of a family of good kernels.

The importance of good kernels is highlighted by their use in connection 
with convolutions.

\(^6\)In the limit, a family of good kernels represents the “Dirac delta function.” This 
terminology comes from physics.
Theorem 4.1 Let \( \{K_n\}_{n=1}^{\infty} \) be a family of good kernels, and \( f \) an integrable function on the circle. Then

\[
\lim_{n \to \infty} (f \ast K_n)(x) = f(x)
\]

whenever \( f \) is continuous at \( x \). If \( f \) is continuous everywhere, then the above limit is uniform.

Because of this result, the family \( \{K_n\} \) is sometimes referred to as an approximation to the identity.

We have previously interpreted convolutions as weighted averages. In this context, the convolution

\[
(f \ast K_n)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-y)K_n(y) \, dy
\]

is the average of \( f(x-y) \), where the weights are given by \( K_n(y) \). However, the weight distribution \( K_n \) concentrates its mass at \( y = 0 \) as \( n \) becomes large. Hence in the integral, the value \( f(x) \) is assigned the full mass as \( n \to \infty \). Figure 4 (b) illustrates this point.

Proof of Theorem 4.1. If \( \epsilon > 0 \) and \( f \) is continuous at \( x \), choose \( \delta \) so that \( |y| < \delta \) implies \( |f(x-y) - f(x)| < \epsilon \). Then, by the first property of good kernels, we can write

\[
(f \ast K_n)(x) - f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(y)f(x-y) \, dy - f(x)
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(y)[f(x-y) - f(x)] \, dy.
\]
Hence,
\[
|(f \ast K_n)(x) - f(x)| = \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(y) [f(x - y) - f(x)] \, dy \right|
\leq \frac{1}{2\pi} \int_{|y|<\delta} |K_n(y)| |f(x - y) - f(x)| \, dy \\
+ \frac{1}{2\pi} \int_{\delta \leq |y| \leq \pi} |K_n(y)| |f(x - y) - f(x)| \, dy \\
\leq \frac{\epsilon}{2\pi} \int_{-\pi}^{\pi} |K_n(y)| \, dy + \frac{2B}{2\pi} \int_{\delta \leq |y| \leq \pi} |K_n(y)| \, dy,
\]
where \( B \) is a bound for \( f \). The first term is bounded by \( \epsilon M/2\pi \) because of the second property of good kernels. By the third property we see that for all large \( n \), the second term will be less than \( \epsilon \). Therefore, for some constant \( C > 0 \) and all large \( n \) we have
\[
|(f \ast K_n)(x) - f(x)| \leq C\epsilon,
\]
thereby proving the first assertion in the theorem. If \( f \) is continuous everywhere, then it is uniformly continuous, and \( \delta \) can be chosen independent of \( x \). This provides the desired conclusion that \( f \ast K_n \to f \) uniformly.

Recall from the beginning of Section 3 that
\[
S_N(f)(x) = (f \ast D_N)(x),
\]
where \( D_N(x) = \sum_{n=-N}^{N} e^{inx} \) is the Dirichlet kernel. It is natural now for us to ask whether \( D_N \) is a good kernel, since if this were true, Theorem 4.1 would imply that the Fourier series of \( f \) converges to \( f(x) \) whenever \( f \) is continuous at \( x \). Unfortunately, this is not the case. Indeed, an estimate shows that \( D_N \) violates the second property; more precisely, one has (see Problem 2)
\[
\int_{-\pi}^{\pi} |D_N(x)| \, dx \geq c \log N, \quad \text{as } N \to \infty.
\]
However, we should note that the formula for \( D_N \) as a sum of exponentials immediately gives
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} D_N(x) \, dx = 1,
\]
so the first property of good kernels is actually verified. The fact that the mean value of \( D_N \) is 1, while the integral of its absolute value is large,
is a result of cancellations. Indeed, Figure 5 shows that the function \( D_N(x) \) takes on positive and negative values and oscillates very rapidly as \( N \) gets large.

![Graph of Dirichlet kernel for large \( N \)](image)

**Figure 5.** The Dirichlet kernel for large \( N \)

This observation suggests that the pointwise convergence of Fourier series is intricate, and may even fail at points of continuity. This is indeed the case, as we will see in the next chapter.

## 5 Cesàro and Abel summability: applications to Fourier series

Since a Fourier series may fail to converge at individual points, we are led to try to overcome this failure by interpreting the limit

\[
\lim_{N \to \infty} S_N(f) = f
\]

in a different sense.

### 5.1 Cesàro means and summation

We begin by taking ordinary averages of the partial sums, a technique which we now describe in more detail.
Suppose we are given a series of complex numbers
\[ c_0 + c_1 + c_2 + \cdots = \sum_{k=0}^{\infty} c_k. \]

We define the \( n^{th} \) partial sum \( s_n \) by
\[ s_n = \sum_{k=0}^{n} c_k, \]
and say that the series converges to \( s \) if \( \lim_{n \to \infty} s_n = s \). This is the most natural and most commonly used type of "summability." Consider, however, the example of the series
\[ 1 - 1 + 1 - 1 + \cdots = \sum_{k=0}^{\infty} (-1)^k. \]

Its partial sums form the sequence \( \{1, 0, 1, 0, \ldots\} \) which has no limit. Because these partial sums alternate evenly between 1 and 0, one might therefore suggest that 1/2 is the "limit" of the sequence, and hence 1/2 equals the "sum" of that particular series. We give a precise meaning to this by defining the average of the first \( N \) partial sums by
\[ \sigma_N = \frac{s_0 + s_1 + \cdots + s_{N-1}}{N}. \]

The quantity \( \sigma_N \) is called the \( N^{th} \) Cesàro mean\(^7\) of the sequence \( \{s_k\} \) or the \( N^{th} \) Cesàro sum of the series \( \sum_{k=0}^{\infty} c_k \).

If \( \sigma_N \) converges to a limit \( \sigma \) as \( N \) tends to infinity, we say that the series \( \sum c_n \) is Cesàro summable to \( \sigma \). In the case of series of functions, we shall understand the limit in the sense of either pointwise or uniform convergence, depending on the situation.

The reader will have no difficulty checking that in the above example (3), the series is Cesàro summable to 1/2. Moreover, one can show that Cesàro summation is a more inclusive process than convergence. In fact, if a series is convergent to \( s \), then it is also Cesàro summable to the same limit \( s \) (Exercise 12).

### 5.2 Fejér's theorem

An interesting application of Cesàro summability appears in the context of Fourier series.

---

\(^7\)Note that if the series \( \sum_{k=1}^{\infty} c_k \) begins with the term \( k = 1 \), then it is common practice to define \( \sigma_N = (s_1 + \cdots + s_N)/N \). This change of notation has little effect on what follows.
We mentioned earlier that the Dirichlet kernels fail to belong to the family of good kernels. Quite surprisingly, their averages are very well behaved functions, in the sense that they do form a family of good kernels.

To see this, we form the $N^{\text{th}}$ Cesàro mean of the Fourier series, which by definition is

$$
\sigma_N(f)(x) = \frac{S_0(f)(x) + \cdots + S_{N-1}(f)(x)}{N}.
$$

Since $S_n(f) = f \ast D_n$, we find that

$$
\sigma_N(f)(x) = (f \ast F_N)(x),
$$

where $F_N(x)$ is the $N$-th Fejér kernel given by

$$
F_N(x) = \frac{D_0(x) + \cdots + D_{N-1}(x)}{N}.
$$

**Lemma 5.1** We have

$$
F_N(x) = \frac{1}{N} \frac{\sin^2(Nx/2)}{\sin^2(x/2)},
$$

and the Fejér kernel is a good kernel.

The proof of the formula for $F_N$ (a simple application of trigonometric identities) is outlined in Exercise 15. To prove the rest of the lemma, note that $F_N$ is positive and \(\frac{1}{2\pi} \int_{-\pi}^{\pi} F_N(x) \, dx = 1\), in view of the fact that a similar identity holds for the Dirichlet kernels $D_n$. However, $\sin^2(x/2) \geq c_\delta > 0$, if $\delta \leq |x| \leq \pi$, hence $F_N(x) \leq 1/(Nc_\delta)$, from which it follows that

$$
\int_{\delta \leq |x| \leq \pi} |F_N(x)| \, dx \to 0 \quad \text{as } N \to \infty.
$$

Applying Theorem 4.1 to this new family of good kernels yields the following important result.

**Theorem 5.2** If $f$ is integrable on the circle, then the Fourier series of $f$ is Cesàro summable to $f$ at every point of continuity of $f$.

Moreover, if $f$ is continuous on the circle, then the Fourier series of $f$ is uniformly Cesàro summable to $f$.

We may now state two corollaries. The first is a result that we have already established. The second is new, and of fundamental importance.
Corollary 5.3 If \( f \) is integrable on the circle and \( \hat{f}(n) = 0 \) for all \( n \), then \( f = 0 \) at all points of continuity of \( f \).

The proof is immediate since all the partial sums are 0, hence all the Cesàro means are 0.

Corollary 5.4 continuous functions on the circle can be uniformly approximated by trigonometric polynomials.

This means that if \( f \) is continuous on \([-\pi, \pi]\) with \( f(-\pi) = f(\pi) \) and \( \varepsilon > 0 \), then there exists a trigonometric polynomial \( P \) such that

\[
|f(x) - P(x)| < \varepsilon \quad \text{for all} \quad -\pi \leq x \leq \pi.
\]

This follows immediately from the theorem since the partial sums, hence the Cesàro means, are trigonometric polynomials. Corollary 5.4 is the periodic analogue of the Weierstrass approximation theorem for polynomials which can be found in Exercise 16.

5.3 Abel means and summation

Another method of summation was first considered by Abel and actually predates the Cesàro method.

A series of complex numbers \( \sum_{k=0}^{\infty} c_k \) is said to be **Abel summable** to \( s \) if for every \( 0 \leq r < 1 \), the series

\[
A(r) = \sum_{k=0}^{\infty} c_k r^k
\]

converges, and

\[
\lim_{r \to 1} A(r) = s.
\]

The quantities \( A(r) \) are called the **Abel means** of the series. One can prove that if the series converges to \( s \), then it is Abel summable to \( s \). Moreover, the method of Abel summability is even more powerful than the Cesàro method: when the series is Cesàro summable, it is always Abel summable to the same sum. However, if we consider the series

\[
1 - 2 + 3 - 4 + 5 - \cdots = \sum_{k=0}^{\infty} (-1)^k (k + 1),
\]

then one can show that it is Abel summable to 1/4 since

\[
A(r) = \sum_{k=0}^{\infty} (-1)^k (k + 1) r^k = \frac{1}{(1 + r)^2},
\]

but this series is not Cesàro summable; see Exercise 13.
5.4 The Poisson kernel and Dirichlet's problem in the unit disc

To adapt Abel summability to the context of Fourier series, we define the Abel means of the function \( f(\theta) \sim \sum_{n=-\infty}^{\infty} a_n e^{in\theta} \) by

\[
A_r(f)(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} a_n e^{in\theta}.
\]

Since the index \( n \) takes positive and negative values, it is natural to write \( c_0 = a_0 \), and \( c_n = a_n e^{in\theta} + a_{-n} e^{-in\theta} \) for \( n > 0 \), so that the Abel means of the Fourier series correspond to the definition given in the previous section for numerical series.

We note that since \( f \) is integrable, \( |a_n| \) is uniformly bounded in \( n \), so that \( A_r(f) \) converges absolutely and uniformly for each \( 0 \leq r < 1 \). Just as in the case of Cesàro means, the key fact is that these Abel means can be written as convolutions

\[
A_r(f)(\theta) = (f * P_r)(\theta),
\]

where \( P_r(\theta) \) is the Poisson kernel given by

\[
P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}.
\]

In fact,

\[
A_r(f)(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} a_n e^{in\theta}
\]

\[
= \sum_{n=-\infty}^{\infty} r^{|n|} \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\varphi) e^{-in\varphi} \, d\varphi \right) e^{in\theta}
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\varphi) \left( \sum_{n=-\infty}^{\infty} r^{|n|} e^{-in(\varphi-\theta)} \right) \, d\varphi,
\]

where the interchange of the integral and infinite sum is justified by the uniform convergence of the series.

**Lemma 5.5** If \( 0 \leq r < 1 \), then

\[
P_r(\theta) = \frac{1 - r^2}{1 - 2r \cos \theta + r^2}.
\]
The Poisson kernel is a good kernel, as \( r \) tends to 1 from below.

**Proof.** The identity \( P_r(\theta) = \frac{1 - r^2}{1 - 2r \cos \theta + r^2} \) has already been derived in Section 1.1. Note that

\[
1 - 2r \cos \theta + r^2 = (1 - r)^2 + 2r(1 - \cos \theta).
\]

Hence if \( 1/2 \leq r \leq 1 \) and \( \delta \leq |\theta| \leq \pi \), then

\[
1 - 2r \cos \theta + r^2 \geq c_\delta > 0.
\]

Thus \( P_r(\theta) \leq (1 - r^2)/c_\delta \) when \( \delta \leq |\theta| \leq \pi \), and the third property of good kernels is verified. Clearly \( P_r(\theta) \geq 0 \), and integrating the expression (4) term by term (which is justified by the absolute convergence of the series) yields

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta) \, d\theta = 1,
\]

thereby concluding the proof that \( P_r \) is a good kernel.

Combining this lemma with Theorem 4.1, we obtain our next result.

**Theorem 5.6** The Fourier series of an integrable function on the circle is Abel summable to \( f \) at every point of continuity. Moreover, if \( f \) is continuous on the circle, then the Fourier series of \( f \) is uniformly Abel summable to \( f \).

We now return to a problem discussed in Chapter 1, where we sketched the solution of the steady-state heat equation \( \Delta u = 0 \) in the unit disc with boundary condition \( u = f \) on the circle. We expressed the Laplacian in terms of polar coordinates, separated variables, and expected that a solution was given by

\[
(5) \quad u(r, \theta) = \sum_{m = -\infty}^{\infty} a_m r^{|m|} e^{i m \theta},
\]

where \( a_m \) was the \( m^{th} \) Fourier coefficient of \( f \). In other words, we were led to take

\[
u(r, \theta) = A_r(f)(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\varphi) P_r(\theta - \varphi) \, d\varphi.
\]

We are now in a position to show that this is indeed the case.

---

\(^8\)In this case, the family of kernels is indexed by a continuous parameter \( 0 \leq r < 1 \), rather than the discrete \( n \) considered previously. In the definition of good kernels, we simply replace \( n \) by \( r \) and take the limit in property (c) appropriately, for example \( r \to 1 \) in this case.
Theorem 5.7 Let $f$ be an integrable function defined on the unit circle. Then the function $u$ defined in the unit disc by the Poisson integral

\begin{equation}
(6) \quad u(r, \theta) = (f * P_r)(\theta)
\end{equation}

has the following properties:

(i) $u$ has two continuous derivatives in the unit disc and satisfies $\Delta u = 0$.

(ii) If $\theta$ is any point of continuity of $f$, then

\[ \lim_{r \to 1} u(r, \theta) = f(\theta). \]

If $f$ is continuous everywhere, then this limit is uniform.

(iii) If $f$ is continuous, then $u(r, \theta)$ is the unique solution to the steady-state heat equation in the disc which satisfies conditions (i) and (ii).

**Proof.** For (i), we recall that the function $u$ is given by the series (5). Fix $\rho < 1$; inside each disc of radius $r < \rho < 1$ centered at the origin, the series for $u$ can be differentiated term by term, and the differentiated series is uniformly and absolutely convergent. Thus $u$ can be differentiated twice (in fact infinitely many times), and since this holds for all $\rho < 1$, we conclude that $u$ is twice differentiable inside the unit disc. Moreover, in polar coordinates,

\[ \Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}, \]

so term by term differentiation shows that $\Delta u = 0$.

The proof of (ii) is a simple application of the previous theorem. To prove (iii) we argue as follows. Suppose $v$ solves the steady-state heat equation in the disc and converges to $f$ uniformly as $r$ tends to 1 from below. For each fixed $r$ with $0 < r < 1$, the function $v(r, \theta)$ has a Fourier series

\[ \sum_{n=-\infty}^{\infty} a_n(r) e^{in\theta} \quad \text{where} \quad a_n(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} v(r, \theta) e^{-in\theta} d\theta. \]

Taking into account that $v(r, \theta)$ solves the equation

\begin{equation}
(7) \quad \frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} + \frac{1}{r^2} \frac{\partial^2 v}{\partial \theta^2} = 0,
\end{equation}
we find that

\begin{equation}
\frac{d^2}{d\theta^2} v(r, \theta) = -\frac{n^2}{r^2} a_n(r).
\end{equation}

Indeed, we may first multiply (7) by $e^{-in\theta}$ and integrate in $\theta$. Then, since $v$ is periodic, two integrations by parts give

\begin{equation}
\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\partial^2 v}{\partial \theta^2}(r, \theta) e^{-in\theta} d\theta = -n^2 a_n(r).
\end{equation}

Finally, we may interchange the order of differentiation and integration, which is permissible since $v$ has two continuous derivatives; this yields (8).

Therefore, we must have $a_n(r) = A_n r^n + B_n r^{-n}$ for some constants $A_n$ and $B_n$, when $n \neq 0$ (see Exercise 11 in Chapter 1). To evaluate the constants, we first observe that each term $a_n(r)$ is bounded because $v$ is bounded, therefore $B_n = 0$. To find $A_n$ we let $r \to 1$. Since $v$ converges uniformly to $f$ as $r \to 1$ we find that

\begin{equation}
A_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-in\theta} d\theta.
\end{equation}

By a similar argument, this formula also holds when $n = 0$. Our conclusion is that for each $0 < r < 1$, the Fourier series of $v$ is given by the series of $u(r, \theta)$, so by the uniqueness of Fourier series for continuous functions, we must have $u = v$.

Remark. By part (iii) of the theorem, we may conclude that if $u$ solves $\Delta u = 0$ in the disc, and converges to 0 uniformly as $r \to 1$, then $u$ must be identically 0. However, if uniform convergence is replaced by pointwise convergence, this conclusion may fail; see Exercise 18.

6 Exercises

1. Suppose $f$ is $2\pi$-periodic and integrable on any finite interval. Prove that if $a, b \in \mathbb{R}$, then

\begin{equation}
\int_a^b f(x) \, dx = \int_{a+2\pi}^{b+2\pi} f(x) \, dx = \int_{a-2\pi}^{b-2\pi} f(x) \, dx.
\end{equation}

Also prove that

\begin{equation}
\int_{-\pi}^{\pi} f(x + a) \, dx = \int_{-\pi}^{\pi} f(x) \, dx = \int_{-\pi+a}^{\pi+a} f(x) \, dx.
\end{equation}
2. In this exercise we show how the symmetries of a function imply certain properties of its Fourier coefficients. Let \( f \) be a \( 2\pi \)-periodic Riemann integrable function defined on \( \mathbb{R} \).

(a) Show that the Fourier series of the function \( f \) can be written as

\[
f(\theta) \sim \hat{f}(0) + \sum_{n \geq 1} [\hat{f}(n) + \hat{f}(-n)] \cos n\theta + i[\hat{f}(n) - \hat{f}(-n)] \sin n\theta.
\]

(b) Prove that if \( f \) is even, then \( \hat{f}(n) = \hat{f}(-n) \), and we get a cosine series.

(c) Prove that if \( f \) is odd, then \( \hat{f}(n) = -\hat{f}(-n) \), and we get a sine series.

(d) Suppose that \( f(\theta + \pi) = f(\theta) \) for all \( \theta \in \mathbb{R} \). Show that \( \hat{f}(n) = 0 \) for all odd \( n \).

(e) Show that \( f \) is real-valued if and only if \( \hat{f}(n) = \hat{f}(-n) \) for all \( n \).

3. We return to the problem of the plucked string discussed in Chapter 1. Show that the initial condition \( f \) is equal to its Fourier sine series

\[
f(x) = \sum_{m=1}^{\infty} A_m \sin mx \quad \text{with} \quad A_m = \frac{2h}{m^2} \frac{\sin mp}{p(\pi - p)}.
\]

[Hint: Note that \( |A_m| \leq C/m^2 \).

4. Consider the \( 2\pi \)-periodic odd function defined on \( [0, \pi] \) by \( f(\theta) = \theta (\pi - \theta) \).

(a) Draw the graph of \( f \).

(b) Compute the Fourier coefficients of \( f \), and show that

\[
f(\theta) = \frac{8}{\pi} \sum_{k \, \text{odd} \geq 1} \frac{\sin k\theta}{k^3}.
\]

5. On the interval \( [-\pi, \pi] \) consider the function

\[
f(\theta) = \begin{cases} 
0 & \text{if } |\theta| > \delta, \\
1 - |\theta|/\delta & \text{if } |\theta| \leq \delta.
\end{cases}
\]

Thus the graph of \( f \) has the shape of a triangular tent. Show that

\[
f(\theta) = \frac{\delta}{2\pi} + 2 \sum_{n=1}^{\infty} \frac{1 - \cos n\delta}{n^2 \pi \delta} \cos n\theta.
\]

6. Let \( f \) be the function defined on \( [-\pi, \pi] \) by \( f(\theta) = |\theta| \).
(a) Draw the graph of $f$.

(b) Calculate the Fourier coefficients of $f$, and show that

$$
\hat{f}(n) = \begin{cases} 
\frac{\pi}{2} & \text{if } n = 0, \\
-1 + (-1)^n & \text{if } n \neq 0.
\end{cases}
$$

(c) What is the Fourier series of $f$ in terms of sines and cosines?

(d) Taking $\theta = 0$, prove that

$$
\sum_{n \text{ odd } \geq 1} \frac{1}{n^2} = \frac{\pi^2}{8} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.
$$

See also Example 2 in Section 1.1.

7. Suppose $\{a_n\}_{n=1}^{N}$ and $\{b_n\}_{n=1}^{N}$ are two finite sequences of complex numbers. Let $B_k = \sum_{n=1}^{k} b_n$ denote the partial sums of the series $\sum b_n$ with the convention $B_0 = 0$.

(a) Prove the summation by parts formula

$$
\sum_{n=M}^{N} a_n b_n = a_N B_N - a_M B_{M-1} - \sum_{n=M}^{N-1} (a_{n+1} - a_n) B_n.
$$

(b) Deduce from this formula Dirichlet’s test for convergence of a series: if the partial sums of the series $\sum b_n$ are bounded, and $\{a_n\}$ is a sequence of real numbers that decreases monotonically to 0, then $\sum a_n b_n$ converges.

8. Verify that $\frac{1}{2i} \sum_{n \neq 0} \frac{e^{inx}}{n}$ is the Fourier series of the $2\pi$-periodic sawtooth function illustrated in Figure 6, defined by $f(0) = 0$, and

$$
f(x) = \begin{cases} 
-\frac{\pi}{2} - \frac{x}{2} & \text{if } -\pi < x < 0, \\
\frac{\pi}{2} - \frac{x}{2} & \text{if } 0 < x < \pi.
\end{cases}
$$

Note that this function is not continuous. Show that nevertheless, the series converges for every $x$ (by which we mean, as usual, that the symmetric partial sums of the series converge). In particular, the value of the series at the origin, namely 0, is the average of the values of $f(x)$ as $x$ approaches the origin from the left and the right.
9. Let \( f(x) = \chi_{[a,b]}(x) \) be the characteristic function of the interval \([a, b] \subset [-\pi, \pi]\), that is,

\[
\chi_{[a,b]}(x) = \begin{cases} 
1 & \text{if } x \in [a, b], \\
0 & \text{otherwise}.
\end{cases}
\]

(a) Show that the Fourier series of \( f \) is given by

\[
f(x) \sim \frac{b-a}{2\pi} + \sum_{n \neq 0} \frac{e^{-ina} - e^{-inb}}{2\pi in} e^{inx}.
\]

The sum extends over all positive and negative integers excluding 0.

(b) Show that if \( a \neq -\pi \) or \( b \neq \pi \) and \( a \neq b \), then the Fourier series does not converge absolutely for any \( x \). [Hint: It suffices to prove that for many values of \( n \) one has \(|\sin n\theta_0| \geq c > 0 \) where \( \theta_0 = (b - a)/2 \).]

(c) However, prove that the Fourier series converges at every point \( x \). What happens if \( a = -\pi \) and \( b = \pi \)?

10. Suppose \( f \) is a periodic function of period \( 2\pi \) which belongs to the class \( C^k \). Show that

\[
\hat{f}(n) = O(1/|n|^k) \quad \text{as } |n| \to \infty.
\]

This notation means that there exists a constant \( C \) such \(|\hat{f}(n)| \leq C/|n|^k\). We could also write this as \(|n|^k \hat{f}(n) = O(1)\), where \( O(1) \) means bounded.

[Hint: Integrate by parts.]

11. Suppose that \( \{f_k\}_{k=1}^{\infty} \) is a sequence of Riemann integrable functions on the interval \([0, 1]\) such that

\[
\int_0^1 |f_k(x) - f(x)| \, dx \to 0 \quad \text{as } k \to \infty.
\]
Show that \( \hat{f}_k(n) \to \hat{f}(n) \) uniformly in \( n \) as \( k \to \infty \).

12. Prove that if a series of complex numbers \( \sum c_n \) converges to \( s \), then \( \sum c_n \) is Cesàro summable to \( s \).
   [Hint: Assume \( s_n \to 0 \) as \( n \to \infty \).]

13. The purpose of this exercise is to prove that Abel summability is stronger than the standard or Cesàro methods of summation.
   
   (a) Show that if the series \( \sum_{n=1}^{\infty} c_n \) of complex numbers converges to a finite limit \( s \), then the series is Abel summable to \( s \). [Hint: Why is it enough to prove the theorem when \( s = 0 \)? Assuming \( s = 0 \), show that if \( s_N = c_1 + \cdots + c_N \), then \( \sum_{n=1}^{N} c_n r^n = (1 - r) \sum_{n=1}^{N} s_n r^n + s_N r^{N+1} \). Let \( N \to \infty \) to show that

   \[
   \sum c_n r^n = (1 - r) \sum s_n r^n.
   \]

   Finally, prove that the right-hand side converges to 0 as \( r \to 1 \).]

   (b) However, show that there exist series which are Abel summable, but that do not converge. [Hint: Try \( c_n = (-1)^n \). What is the Abel limit of \( \sum c_n \)?]

   (c) Argue similarly to prove that if a series \( \sum_{n=1}^{\infty} c_n \) is Cesàro summable to \( \sigma \), then it is Abel summable to \( \sigma \). [Hint: Note that

   \[
   \sum_{n=1}^{\infty} c_n r^n = (1 - r)^2 \sum_{n=1}^{\infty} n \sigma_n r^n,
   \]

   and assume \( \sigma = 0 \).]

   (d) Give an example of a series that is Abel summable but not Cesàro summable.
   [Hint: Try \( c_n = (-1)^{n-1} n \). Note that if \( \sum c_n \) is Cesàro summable, then \( c_n/n \) tends to 0.]

The results above can be summarized by the following implications about series:

\[
\text{convergent} \implies \text{Cesàro summable} \implies \text{Abel summable},
\]

and the fact that none of the arrows can be reversed.

14. This exercise deals with a theorem of Tauber which says that under an additional condition on the coefficients \( c_n \), the above arrows can be reversed.

   (a) If \( \sum c_n \) is Cesàro summable to \( \sigma \) and \( c_n = o(1/n) \) (that is, \( n c_n \to 0 \)), then \( \sum c_n \) converges to \( \sigma \). [Hint: \( s_n - \sigma_n = [(n - 1)c_n + \cdots + c_2]/n \).]

   (b) The above statement holds if we replace Cesàro summable by Abel summable.
   [Hint: Estimate the difference between \( \sum_{n=1}^{N} c_n \) and \( \sum_{n=1}^{N} c_n r^n \) where \( r = 1 - 1/N \).]
15. Prove that the Fejér kernel is given by

\[ F_N(x) = \frac{1}{N} \frac{\sin^2(Nx/2)}{\sin^2(x/2)}. \]

[Hint: Remember that \( NF_N(x) = D_0(x) + \cdots + D_{N-1}(x) \) where \( D_n(x) \) is the Dirichlet kernel. Therefore, if \( \omega = e^{ix} \) we have

\[ NF_N(x) = \sum_{n=0}^{N-1} \frac{\omega^{-n} - \omega^{n+1}}{1 - \omega}. \]

16. The Weierstrass approximation theorem states: Let \( f \) be a continuous function on the closed and bounded interval \([a, b] \subset \mathbb{R}\). Then, for any \( \epsilon > 0 \), there exists a polynomial \( P \) such that

\[ \sup_{x \in [a, b]} |f(x) - P(x)| < \epsilon. \]

Prove this by applying Corollary 5.4 of Fejér’s theorem and using the fact that the exponential function \( e^{ix} \) can be approximated by polynomials uniformly on any interval.

17. In Section 5.4 we proved that the Abel means of \( f \) converge to \( f \) at all points of continuity, that is,

\[ \lim_{r \to 1} A_r(f)(\theta) = \lim_{r \to 1} (P_r * f)(\theta) = f(\theta), \quad \text{with } 0 < r < 1, \]

whenever \( f \) is continuous at \( \theta \). In this exercise, we will study the behavior of \( A_r(f)(\theta) \) at certain points of discontinuity.

An integrable function is said to have a **jump discontinuity** at \( \theta \) if the two limits

\[ \lim_{h \to 0} f(\theta + h) = f(\theta^+) \quad \text{and} \quad \lim_{h \to 0} f(\theta - h) = f(\theta^-) \]

exist.

(a) Prove that if \( f \) has a jump discontinuity at \( \theta \), then

\[ \lim_{r \to 1} A_r(f)(\theta) = \frac{f(\theta^+) + f(\theta^-)}{2}, \quad \text{with } 0 \leq r < 1. \]

[Hint: Explain why \( \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta) d\theta = \frac{1}{2\pi} \int_{0}^{\pi} P_r(\theta) d\theta = \frac{1}{2} \), then modify the proof given in the text.]
(b) Using a similar argument, show that if \( f \) has a jump discontinuity at \( \theta \), the Fourier series of \( f \) at \( \theta \) is Cesàro summable to \( \frac{f(\theta^+)+f(\theta^-)}{2} \).

18. If \( P_r(\theta) \) denotes the Poisson kernel, show that the function

\[
u(r, \theta) = \frac{\partial P_r}{\partial \theta},\]

defined for \( 0 \leq r < 1 \) and \( \theta \in \mathbb{R} \), satisfies:

(i) \( \Delta u = 0 \) in the disc.

(ii) \( \lim_{r \to 1} u(r, \theta) = 0 \) for each \( \theta \).

However, \( u \) is not identically zero.

19. Solve Laplace's equation \( \Delta u = 0 \) in the semi infinite strip

\[S = \{(x, y) : 0 < x < 1, 0 < y\},\]

subject to the following boundary conditions

\[
\begin{aligned}
    u(0, y) &= 0 & \text{when } 0 \leq y, \\
    u(1, y) &= 0 & \text{when } 0 \leq y, \\
    u(x, 0) &= f(x) & \text{when } 0 \leq x \leq 1
\end{aligned}
\]

where \( f \) is a given function, with of course \( f(0) = f(1) = 0 \). Write

\[f(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x)\]

and expand the general solution in terms of the special solutions given by

\[u_n(x, y) = e^{-n\pi y} \sin(n\pi x).\]

Express \( u \) as an integral involving \( f \), analogous to the Poisson integral formula (6).

20. Consider the Dirichlet problem in the annulus defined by \( \{(r, \theta) : \rho < r < 1\} \), where \( 0 < \rho < 1 \) is the inner radius. The problem is to solve

\[
\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0
\]

subject to the boundary conditions

\[
\begin{aligned}
    u(1, \theta) &= f(\theta), \\
    u(\rho, \theta) &= g(\theta),
\end{aligned}
\]
where \( f \) and \( g \) are given continuous functions.

Arguing as we have previously for the Dirichlet problem in the disc, we can hope to write

\[
u(r, \theta) = \sum c_n(r)e^{in\theta}
\]

with \( c_n(r) = A_n r^n + B_n r^{-n}, \ n \neq 0. \) Set

\[
f(\theta) \sim \sum a_n e^{in\theta} \quad \text{and} \quad g(\theta) \sim \sum b_n e^{in\theta}.
\]

We want \( c_n(1) = a_n \) and \( c_n(\rho) = b_n. \) This leads to the solution

\[
u(r, \theta) = \sum_{n \neq 0} \left( \frac{1}{\rho^n - \rho^{-n}} \right) \left[ ((\rho/r)^n - (r/\rho)^n) a_n + (r^n - r^{-n}) b_n \right] e^{in\theta} + a_0 + (b_0 - a_0) \frac{\log r}{\log \rho}.
\]

Show that as a result we have

\[
u(r, \theta) - (P_r \ast f)(\theta) \to 0 \quad \text{as } r \to 1 \text{ uniformly in } \theta,
\]

and

\[
u(r, \theta) - (P_\rho \ast g)(\theta) \to 0 \quad \text{as } r \to \rho \text{ uniformly in } \theta.
\]

7 Problems

1. One can construct Riemann integrable functions on \([0, 1]\) that have a dense set of discontinuities as follows.

(a) Let \( f(x) = 0 \) when \( x < 0, \) and \( f(x) = 1 \) if \( x \geq 0. \) Choose a countable dense sequence \( \{r_n\} \) in \([0, 1]. \) Then, show that the function

\[
F(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} f(x - r_n)
\]

is integrable and has discontinuities at all points of the sequence \( \{r_n\}. \)

[Hint: \( F \) is monotonic and bounded.]

(b) Consider next

\[
F(x) = \sum_{n=1}^{\infty} 3^{-n} g(x - r_n),
\]

where \( g(x) = \sin 1/x \) when \( x \neq 0, \) and \( g(0) = 0. \) Then \( F \) is integrable, discontinuous at each \( x = r_n, \) and fails to be monotonic in any subinterval of \([0, 1]. \)

[Hint: Use the fact that \( 3^{-k} > \sum_{n>k} 3^{-n}. \)]
(c) The original example of Riemann is the function

\[ F(x) = \sum_{n=1}^{\infty} \frac{(nx)}{n^2}, \]

where \( (x) = x \) for \( x \in (-1/2, 1/2] \) and \( (x) \) is continued to \( \mathbb{R} \) by periodicity, that is, \( (x + 1) = (x) \). It can be shown that \( F \) is discontinuous whenever \( x = m/2n \), where \( m, n \in \mathbb{Z} \) with \( m \) odd and \( n \neq 0 \).

2. Let \( D_N \) denote the Dirichlet kernel

\[ D_N(\theta) = \sum_{k=-N}^{N} e^{ik\theta} = \frac{\sin((N + 1/2)\theta)}{\sin(\theta/2)}, \]

and define

\[ L_N = \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_N(\theta)| \, d\theta. \]

(a) Prove that

\[ L_N \geq c \log N \]

for some constant \( c > 0 \). [Hint: Show that \( |D_N(\theta)| \geq c \frac{\sin((N+1/2)\theta)}{\theta} \), change variables, and prove that

\[ L_N \geq c \int_{\pi}^{N\pi} \frac{\sin |\theta|}{|\theta|} \, d\theta + O(1). \]

Write the integral as a sum \( \sum_{k=1}^{N-1} \int_{k\pi}^{(k+1)\pi} \). To conclude, use the fact that \( \sum_{k=1}^{n} 1/k \geq c \log n \).] A more careful estimate gives

\[ L_N = \frac{4}{\pi^2} \log N + O(1). \]

(b) Prove the following as a consequence: for each \( n \geq 1 \), there exists a continuous function \( f_n \) such that \( |f_n| \leq 1 \) and \( |S_n(f_n)(0)| \geq c' \log n \). [Hint: The function \( g_n \), which is equal to 1 when \( D_n \) is positive and -1 when \( D_n \) is negative has the desired property but is not continuous. Approximate \( g_n \) in the integral norm (in the sense of Lemma 3.2) by continuous functions \( h_k \) satisfying \( |h_k| \leq 1 \).]

3. Littlewood provided a refinement of Tauber’s theorem:
7. Problems

(a) If \( \sum c_n \) is Abel summable to \( s \) and \( c_n = O(1/n) \), then \( \sum c_n \) converges to \( s \).

(b) As a consequence, if \( \sum c_n \) is Cesàro summable to \( s \) and \( c_n = O(1/n) \), then \( \sum c_n \) converges to \( s \).

These results may be applied to Fourier series. By Exercise 17, they imply that if \( f \) is an integrable function that satisfies \( \hat{f}(\nu) = O(1/|\nu|) \), then:

(i) If \( f \) is continuous at \( \theta \), then

\[
S_N(f)(\theta) \to f(\theta) \quad \text{as } N \to \infty.
\]

(ii) If \( f \) has a jump discontinuity at \( \theta \), then

\[
S_N(f)(\theta) \to \frac{f(\theta^+) + f(\theta^-)}{2} \quad \text{as } N \to \infty.
\]

(iii) If \( f \) is continuous on \([-\pi, \pi]\), then \( S_N(f) \to f \) uniformly.

For the simpler assertion (b), hence a proof of (i), (ii), and (iii), see Problem 5 in Chapter 4.
3 Convergence of Fourier Series

The sine and cosine series, by which one can represent an arbitrary function in a given interval, enjoy among other remarkable properties that of being convergent. This property did not escape the great geometer (Fourier) who began, through the introduction of the representation of functions just mentioned, a new career for the applications of analysis; it was stated in the Memoir which contains his first research on heat. But no one so far, to my knowledge, gave a general proof of it . . .

G. Dirichlet, 1829

In this chapter, we continue our study of the problem of convergence of Fourier series. We approach the problem from two different points of view.

The first is "global" and concerns the overall behavior of a function $f$ over the entire interval $[0, 2\pi]$. The result we have in mind is "mean-square convergence": if $f$ is integrable on the circle, then

$$\frac{1}{2\pi} \int_0^{2\pi} |f(\theta) - S_N(f)(\theta)|^2 \, d\theta \to 0 \quad \text{as } N \to \infty.$$ 

At the heart of this result is the fundamental notion of "orthogonality"; this idea is expressed in terms of vector spaces with inner products, and their related infinite dimensional variants, the Hilbert spaces. A connected result is the Parseval identity which equates the mean-square "norm" of the function with a corresponding norm of its Fourier coefficients. Orthogonality is a fundamental mathematical notion which has many applications in analysis.

The second viewpoint is "local" and concerns the behavior of $f$ near a given point. The main question we consider is the problem of pointwise convergence: does the Fourier series of $f$ converge to the value $f(\theta)$ for a given $\theta$? We first show that this convergence does indeed hold whenever $f$ is differentiable at $\theta$. As a corollary, we obtain the Riemann localization principle, which states that the question of whether or not $S_N(f)(\theta) \to f(\theta)$ is completely determined by the behavior of $f$ in an
arbitrarily small interval about $\theta$. This is a remarkable result since the Fourier coefficients, hence the Fourier series, of $f$ depend on the values of $f$ on the whole interval $[0, 2\pi]$.

Even tough convergence of the Fourier series holds at points where $f$ is differentiable, it may fail if $f$ is merely continuous. The chapter concludes with the presentation of a continuous function whose Fourier series does not converge at a given point, as promised earlier.

1 Mean-square convergence of Fourier series

The aim of this section is the proof of the following theorem.

**Theorem 1.1** Suppose $f$ is integrable on the circle. Then

$$\frac{1}{2\pi} \int_0^{2\pi} |f(\theta) - S_N(f)(\theta)|^2 \, d\theta \rightarrow 0 \quad \text{as } N \rightarrow \infty.$$ 

As we remarked earlier, the key concept involved is that of orthogonality. The correct setting for orthogonality is in a vector space equipped with an inner product.

1.1 Vector spaces and inner products

We now review the definitions of a vector space over $\mathbb{R}$ or $\mathbb{C}$, an inner product, and its associated norm. In addition to the familiar finite-dimensional vector spaces $\mathbb{R}^d$ and $\mathbb{C}^d$, we also examine two infinite-dimensional examples which play a central role in the proof of Theorem 1.1.

Preliminaries on vector spaces

A vector space $V$ over the real numbers $\mathbb{R}$ is a set whose elements may be "added" together, and "multiplied" by scalars. More precisely, we may associate to any pair $X, Y \in V$ an element in $V$ called their sum and denoted by $X + Y$. We require that this addition respects the usual laws of arithmetic, such as commutativity $X + Y = Y + X$, and associativity $X + (Y + Z) = (X + Y) + Z$, etc. Also, given any $X \in V$ and real number $\lambda$, we assign an element $\lambda X \in V$ called the product of $X$ by $\lambda$. This scalar multiplication must satisfy the standard properties, for instance $\lambda_1(\lambda_2 X) = (\lambda_1 \lambda_2)X$ and $\lambda(X + Y) = \lambda X + \lambda Y$. We may instead allow scalar multiplication by numbers in $\mathbb{C}$; we then say that $V$ is a vector space over the complex numbers.
For example, the set $\mathbb{R}^d$ of $d$-tuples of real numbers $(x_1, x_2, \ldots, x_d)$ is a vector space over the reals. Addition is defined componentwise by

$$(x_1, \ldots, x_d) + (y_1, \ldots, y_d) = (x_1 + y_1, \ldots, x_d + y_d),$$

and so is multiplication by a scalar $\lambda \in \mathbb{R}$:

$$\lambda(x_1, \ldots, x_d) = (\lambda x_1, \ldots, \lambda x_d).$$

Similarly, the space $\mathbb{C}^d$ (the complex version of the previous example) is the set of $d$-tuples of complex numbers $(z_1, z_2, \ldots, z_d)$. It is a vector space over $\mathbb{C}$ with addition defined componentwise by

$$(z_1, \ldots, z_d) + (w_1, \ldots, w_d) = (z_1 + w_1, \ldots, z_d + w_d).$$

Multiplication by scalars $\lambda \in \mathbb{C}$ is given by

$$\lambda(z_1, \ldots, z_d) = (\lambda z_1, \ldots, \lambda z_d).$$

An inner product on a vector space $V$ over $\mathbb{R}$ associates to any pair $X, Y$ of elements in $V$ a real number which we denote by $(X, Y)$. In particular, the inner product must be symmetric $(X, Y) = (Y, X)$ and linear in both variables; that is,

$$(\alpha X + \beta Y, Z) = \alpha(X, Z) + \beta(Y, Z)$$

whenever $\alpha, \beta \in \mathbb{R}$ and $X, Y, Z \in V$. Also, we require that the inner product be positive-definite, that is, $(X, X) \geq 0$ for all $X$ in $V$. In particular, given an inner product $(\cdot, \cdot)$ we may define the norm of $X$ by

$$\|X\| = (X, X)^{1/2}.$$

If in addition $\|X\| = 0$ implies $X = 0$, we say that the inner product is strictly positive-definite.

For example, the space $\mathbb{R}^d$ is equipped with a (strictly positive-definite) inner product defined by

$$(X, Y) = x_1y_1 + \cdots + x_dy_d$$

when $X = (x_1, \ldots, x_d)$ and $Y = (y_1, \ldots, y_d)$. Then

$$\|X\| = (X, X)^{1/2} = \sqrt{x_1^2 + \cdots + x_d^2},$$
which is the usual Euclidean distance. One also uses the notation $|X|$ instead of $\|X\|$.

For vector spaces over the complex numbers, the inner product of two elements is a complex number. Moreover, these inner products are called Hermitian (instead of symmetric) since they must satisfy $(X, Y) = (Y, X)$. Hence the inner product is linear in the first variable, but conjugate-linear in the second:

$$(\alpha X + \beta Y, Z) = \alpha(X, Z) + \beta(Y, Z) \quad \text{and}$$

$$(X, \alpha Y + \beta Z) = \overline{\alpha}(X, Y) + \overline{\beta}(X, Z).$$

Also, we must have $(X, X) \geq 0$, and the norm of $X$ is defined by $\|X\| = \sqrt{(X, X)}$ as before. Again, the inner product is strictly positive-definite if $\|X\| = 0$ implies $X = 0$.

For example, the inner product of two vectors $Z = (z_1, \ldots, z_d)$ and $W = (w_1, \ldots, w_d)$ in $\mathbb{C}^d$ is defined by

$$(Z, W) = z_1 \overline{w_1} + \cdots + z_d \overline{w_d}.$$  

The norm of the vector $Z$ is then given by

$$\|Z\| = (Z, Z)^{1/2} = \sqrt{|z_1|^2 + \cdots + |z_d|^2}.$$

The presence of an inner product on a vector space allows one to define the geometric notion of “orthogonality.” Let $V$ be a vector space (over $\mathbb{R}$ or $\mathbb{C}$) with inner product $(\cdot, \cdot)$ and associated norm $\|\cdot\|$. Two elements $X$ and $Y$ are orthogonal if $(X, Y) = 0$, and we write $X \perp Y$. Three important results can be derived from this notion of orthogonality:

(i) The Pythagorean theorem: if $X$ and $Y$ are orthogonal, then

$$\|X + Y\|^2 = \|X\|^2 + \|Y\|^2.$$  

(ii) The Cauchy-Schwarz inequality: for any $X, Y \in V$ we have

$$|(X, Y)| \leq \|X\| \|Y\|.$$  

(iii) The triangle inequality: for any $X, Y \in V$ we have

$$\|X + Y\| \leq \|X\| + \|Y\|.$$
The proofs of these facts are simple. For (i) it suffices to expand 
\((X + Y, X + Y)\) and use the assumption that 
\((X, Y) = 0\).

For (ii), we first dispose of the case when \(\|Y\| = 0\) by showing that 
this implies \((X, Y) = 0\) for all \(X\). Indeed, for all real \(t\) we have

\[
0 \leq \|X + tY\|^2 = \|X\|^2 + 2t \text{Re}(X, Y)
\]

and \(\text{Re}(X, Y) \neq 0\) contradicts the inequality if we take \(t\) to be large and 
positive (or negative). Similarly, by considering \(\|X + itY\|^2\), we find that 
\(\text{Im}(X, Y) = 0\).

If \(\|Y\| \neq 0\), we may set \(c = (X, Y)/(Y, Y)\); then \(X - cY\) is orthogonal 
to \(Y\), and therefore also to \(cY\). If we write \(X = X - cY + cY\) and apply 
the Pythagorean theorem, we get

\[
\|X\|^2 = \|X - cY\|^2 + \|cY\|^2 \geq |c|^2\|Y\|^2.
\]

Taking square roots on both sides gives the result. Note that we have 
equality in the above precisely when \(X = cY\).

Finally, for (iii) we first note that

\[
\|X + Y\|^2 = (X, X) + (X, Y) + (Y, X) + (Y, Y).
\]

But \((X, X) = \|X\|^2\), \((Y, Y) = \|Y\|^2\), and by the Cauchy-Schwarz inequality

\[
|(X, Y) + (Y, X)| \leq 2 \|X\| \|Y\|,
\]

therefore

\[
\|X + Y\|^2 \leq \|X\|^2 + 2 \|X\| \|Y\| + \|Y\|^2 = (\|X\| + \|Y\|)^2.
\]

Two important examples

The vector spaces \(\mathbb{R}^d\) and \(\mathbb{C}^d\) are finite dimensional. In the context 
of Fourier series, we need to work with two infinite-dimensional vector 
spaces, which we now describe.

**Example 1.** The vector space \(\ell^2(\mathbb{Z})\) over \(\mathbb{C}\) is the set of all (two-sided) 
infinite sequences of complex numbers

\[
(\ldots, a_{-n}, \ldots, a_{-1}, a_0, a_1, \ldots, a_n, \ldots)
\]

such that

\[
\sum_{n \in \mathbb{Z}} |a_n|^2 < \infty;
\]
that is, the series converges. Addition is defined componentwise, and so is scalar multiplication. The inner product between the two vectors $A = (\ldots, a_{-1}, a_0, a_1, \ldots)$ and $B = (\ldots, b_{-1}, b_0, b_1, \ldots)$ is defined by the absolutely convergent series

$$(A, B) = \sum_{n \in \mathbb{Z}} a_n \overline{b_n}.$$  

The norm of $A$ is then given by

$$\|A\| = (A, A)^{1/2} = \left(\sum_{n \in \mathbb{Z}} |a_n|^2 \right)^{1/2}.$$  

We must first check that $\ell^2(\mathbb{Z})$ is a vector space. This requires that if $A$ and $B$ are two elements in $\ell^2(\mathbb{Z})$, then so is the vector $A + B$. To see this, for each integer $N > 0$ we let $A_N$ denote the truncated element

$$A_N = (\ldots, 0, 0, a_{-N}, \ldots, a_{-1}, a_0, a_1, \ldots, a_N, 0, 0, \ldots),$$

where we have set $a_n = 0$ whenever $|n| > N$. We define the truncated element $B_N$ similarly. Then, by the triangle inequality which holds in a finite dimensional Euclidean space, we have

$$\|A_N + B_N\| \leq \|A_N\| + \|B_N\| \leq \|A\| + \|B\|.$$  

Thus

$$\sum_{|n| \leq N} |a_n + b_n|^2 \leq (\|A\| + \|B\|)^2,$$

and letting $N$ tend to infinity gives $\sum_{n \in \mathbb{Z}} |a_n + b_n|^2 < \infty$. It also follows that $\|A + B\| \leq \|A\| + \|B\|$, which is the triangle inequality. The Cauchy-Schwarz inequality, which states that the sum $\sum_{n \in \mathbb{Z}} a_n \overline{b_n}$ converges absolutely and that $|(A, B)| \leq \|A\| \|B\|$, can be deduced in the same way from its finite analogue.

In the three examples $\mathbb{R}^d$, $\mathbb{C}^d$, and $\ell^2(\mathbb{Z})$, the vector spaces with their inner products and norms satisfy two important properties:

(i) The inner product is strictly positive-definite, that is, $\|X\| = 0$ implies $X = 0$.

(ii) The vector space is complete, which by definition means that every Cauchy sequence in the norm converges to a limit in the vector space.
An inner product space with these two properties is called a **Hilbert space**. We see that $\mathbb{R}^d$ and $\mathbb{C}^d$ are examples of finite-dimensional Hilbert spaces, while $\ell^2(\mathbb{Z})$ is an example of an infinite-dimensional Hilbert space (see Exercises 1 and 2). If either of the conditions above fail, the space is called a **pre-Hilbert space**.

We now give an important example of a pre-Hilbert space where both conditions (i) and (ii) fail.

**Example 2.** Let $\mathcal{R}$ denote the set of complex-valued Riemann integrable functions on $[0, 2\pi]$ (or equivalently, integrable functions on the circle). This is a vector space over $\mathbb{C}$. Addition is defined pointwise by

$$(f + g)(\theta) = f(\theta) + g(\theta).$$

Naturally, multiplication by a scalar $\lambda \in \mathbb{C}$ is given by

$$(\lambda f)(\theta) = \lambda \cdot f(\theta).$$

An inner product is defined on this vector space by

$$(1) \quad (f, g) = \frac{1}{2\pi} \int_0^{2\pi} f(\theta)\overline{g(\theta)} \, d\theta.$$

The norm of $f$ is then

$$\|f\| = \left( \frac{1}{2\pi} \int_0^{2\pi} |f(\theta)|^2 \, d\theta \right)^{1/2}.$$

One needs to check that the analogue of the Cauchy-Schwarz and triangle inequalities hold in this example; that is, $|(f, g)| \leq \|f\| \|g\|$ and $\|f + g\| \leq \|f\| + \|g\|$. While these facts can be obtained as consequences of the corresponding inequalities in the previous examples, the argument is a little elaborate and we prefer to proceed differently.

We first observe that $2AB \leq (A^2 + B^2)$ for any two real numbers $A$ and $B$. If we set $A = \lambda^{1/2}|f(\theta)|$ and $B = \lambda^{-1/2}|g(\theta)|$ with $\lambda > 0$, we get

$$|f(\theta)\overline{g(\theta)}| \leq \frac{1}{2}(\lambda|f(\theta)|^2 + \lambda^{-1}|g(\theta)|^2).$$

We then integrate this in $\theta$ to obtain

$$|(f, g)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(\theta)||\overline{g(\theta)}| \, d\theta \leq \frac{1}{2}(\lambda\|f\|^2 + \lambda^{-1}\|g\|^2).$$

Then, put $\lambda = \|g\|/\|f\|$ to get the Cauchy-Schwarz inequality. The triangle inequality is then a simple consequence, as we have seen above.
Of course, in our choice of λ we must assume that ∥f∥ ≠ 0 and ∥g∥ ≠ 0, which leads us to the following observation.

In ℜ, condition (i) for a Hilbert space fails, since ∥f∥ = 0 implies only that f vanishes at its points of continuity. This is not a very serious problem since in the appendix we show that an integrable function is continuous except for a "negligible" set, so that ∥f∥ = 0 implies that f vanishes except on a set of "measure zero." One can get around the difficulty that f is not identically zero by adopting the convention that such functions are actually the zero function, since for the purpose of integration, f behaves precisely like the zero function.

A more essential difficulty is that the space ℜ is not complete. One way to see this is to start with the function

\[ f(θ) = \begin{cases} 0 & \text{for } θ = 0, \\ \log(1/θ) & \text{for } 0 < θ \leq 2π. \end{cases} \]

Since f is not bounded, it does not belong to the space ℜ. Moreover, the sequence of truncations \( f_n \) defined by

\[ f_n(θ) = \begin{cases} 0 & \text{for } 0 \leq θ \leq 1/n, \\ f(θ) & \text{for } 1/n < θ \leq 2π \end{cases} \]

can easily be seen to form a Cauchy sequence in ℜ (see Exercise 5). However, this sequence cannot converge to an element in ℜ, since that limit, if it existed, would have to be f; for another example, see Exercise 7.

This and more complicated examples motivate the search for the completion of ℜ, the class of Riemann integrable functions on [0, 2π]. The construction and identification of this completion, the Lebesgue class \( L^2([0, 2π]) \), represents an important turning point in the development of analysis (somewhat akin to the much earlier completion of the rationals, that is, the passage from \( \mathbb{Q} \) to \( \mathbb{R} \)). A further discussion of these fundamental ideas will be postponed until Book III, where we take up the Lebesgue theory of integration.

We now turn to the proof of Theorem 1.1.

1.2 Proof of mean-square convergence

Consider the space ℜ of integrable functions on the circle with inner product

\[ (f, g) = \frac{1}{2π} \int_0^{2π} f(θ) \overline{g(θ)} \, dθ \]
and norm \( \|f\| \) defined by

\[
\|f\|^2 = (f, f) = \frac{1}{2\pi} \int_0^{2\pi} |f(\theta)|^2 \, d\theta.
\]

With this notation, we must prove that \( \|f - S_N(f)\| \to 0 \) as \( N \) tends to infinity.

For each integer \( n \), let \( e_n(\theta) = e^{in\theta} \), and observe that the family \( \{e_n\}_{n \in \mathbb{Z}} \) is orthonormal; that is,

\[
(e_n, e_m) = \begin{cases} 
1 & \text{if } n = m \\
0 & \text{if } n \neq m.
\end{cases}
\]

Let \( f \) be an integrable function on the circle, and let \( a_n \) denote its Fourier coefficients. An important observation is that these Fourier coefficients are represented by inner products of \( f \) with the elements in the orthonormal set \( \{e_n\}_{n \in \mathbb{Z}} \):

\[
(f, e_n) = \frac{1}{2\pi} \int_0^{2\pi} f(\theta)e^{-in\theta} \, d\theta = a_n.
\]

In particular, \( S_N(f) = \sum_{|n| \leq N} a_n e_n \). Then the orthonormal property of the family \( \{e_n\} \) and the fact that \( a_n = (f, e_n) \) imply that the difference \( f - \sum_{|n| \leq N} a_n e_n \) is orthogonal to \( e_n \) for all \( |n| \leq N \). Therefore, we must have

\[
(f - \sum_{|n| \leq N} a_n e_n) \perp \sum_{|n| \leq N} b_n e_n
\]

for any complex numbers \( b_n \). We draw two conclusions from this fact.

First, we can apply the Pythagorean theorem to the decomposition

\[
f = f - \sum_{|n| \leq N} a_n e_n + \sum_{|n| \leq N} a_n e_n,
\]

where we now choose \( b_n = a_n \), to obtain

\[
\|f\|^2 = \|f - \sum_{|n| \leq N} a_n e_n\|^2 + \| \sum_{|n| \leq N} a_n e_n \|^2.
\]

Since the orthonormal property of the family \( \{e_n\}_{n \in \mathbb{Z}} \) implies that

\[
\| \sum_{|n| \leq N} a_n e_n \|^2 = \sum_{|n| \leq N} |a_n|^2,
\]
we deduce that
\[
\|f\|^2 = \|f - S_N(f)\|^2 + \sum_{|n| \leq N} |a_n|^2.
\]

The second conclusion we may draw from (2) is the following simple lemma.

**Lemma 1.2 (Best approximation)** If \( f \) is integrable on the circle with Fourier coefficients \( a_n \), then
\[
\|f - S_N(f)\| \leq \|f - \sum_{|n| \leq N} c_n e_n\|
\]
for any complex numbers \( c_n \). Moreover, equality holds precisely when \( c_n = a_n \) for all \( |n| \leq N \).

**Proof.** This follows immediately by applying the Pythagorean theorem to
\[
f - \sum_{|n| \leq N} c_n e_n = f - S_N(f) + \sum_{|n| \leq N} b_n e_n,
\]
where \( b_n = a_n - c_n \).

This lemma has a clear geometric interpretation. It says that the trigonometric polynomial of degree at most \( N \) which is closest to \( f \) in the norm \( \| \cdot \| \) is the partial sum \( S_N(f) \). This geometric property of the partial sums is depicted in Figure 1, where the orthogonal projection of \( f \) in the plane spanned by \( \{e_{-N}, \ldots, e_0, \ldots, e_N\} \) is simply \( S_N(f) \).

![Figure 1. The best approximation lemma](image)

We can now give the proof that \( \|S_N(f) - f\| \to 0 \) using the best approximation lemma, as well as the important fact that trigonometric polynomials are dense in the space of continuous functions on the circle.
Suppose that $f$ is continuous on the circle. Then, given $\epsilon > 0$, there exists (by Corollary 5.4 in Chapter 2) a trigonometric polynomial $P$, say of degree $M$, such that
\[ |f(\theta) - P(\theta)| < \epsilon \quad \text{for all } \theta. \]
In particular, taking squares and integrating this inequality yields $\|f - P\| < \epsilon$, and by the best approximation lemma we conclude that
\[ \|f - S_N(f)\| < \epsilon \quad \text{whenever } N \geq M. \]
This proves Theorem 1.1 when $f$ is continuous.

If $f$ is merely integrable, we can no longer approximate $f$ uniformly by trigonometric polynomials. Instead, we apply the approximation Lemma 3.2 in Chapter 2 and choose a continuous function $g$ on the circle which satisfies
\[ \sup_{\theta \in [0,2\pi]} |g(\theta)| \leq \sup_{\theta \in [0,2\pi]} |f(\theta)| = B, \]
and
\[ \int_0^{2\pi} |f(\theta) - g(\theta)| \, d\theta < \epsilon^2. \]
Then we get
\[ \|f - g\|^2 = \frac{1}{2\pi} \int_0^{2\pi} |f(\theta) - g(\theta)|^2 \, d\theta \]
\[ = \frac{1}{2\pi} \int_0^{2\pi} |f(\theta) - g(\theta)| \cdot |f(\theta) - g(\theta)| \, d\theta \]
\[ \leq \frac{2B}{2\pi} \int_0^{2\pi} |f(\theta) - g(\theta)| \, d\theta \]
\[ \leq C\epsilon^2. \]
Now we may approximate $g$ by a trigonometric polynomial $P$ so that $\|g - P\| < \epsilon$. Then $\|f - P\| < C'\epsilon$, and we may again conclude by applying the best approximation lemma. This completes the proof that the partial sums of the Fourier series of $f$ converge to $f$ in the mean square norm $\| \cdot \|$.

Note that this result and the relation (3) imply that if $a_n$ is the $n^{th}$ Fourier coefficient of an integrable function $f$, then the series $\sum_{n=-\infty}^{\infty} |a_n|^2$ converges, and in fact we have Parseval’s identity
\[ \sum_{n=-\infty}^{\infty} |a_n|^2 = \|f\|^2. \]
This identity provides an important connection between the norms in the two vector spaces \( \ell^2(\mathbb{Z}) \) and \( \mathcal{R} \).

We now summarize the results of this section.

**Theorem 1.3** Let \( f \) be an integrable function on the circle with \( f \sim \sum_{n=-\infty}^{\infty} a_n e^{in\theta} \). Then we have:

(i) **Mean-square convergence of the Fourier series**

\[
\frac{1}{2\pi} \int_{0}^{2\pi} |f(\theta) - S_N(f)(\theta)|^2 \, d\theta \to 0 \quad \text{as} \, N \to \infty.
\]

(ii) **Parseval’s identity**

\[
\sum_{n=-\infty}^{\infty} |a_n|^2 = \frac{1}{2\pi} \int_{0}^{2\pi} |f(\theta)|^2 \, d\theta.
\]

**Remark 1.** If \( \{e_n\} \) is any orthonormal family of functions on the circle, and \( a_n = (f, e_n) \), then we may deduce from the relation (3) that

\[
\sum_{n=-\infty}^{\infty} |a_n|^2 \leq \|f\|^2.
\]

This is known as **Bessel’s inequality**. Equality holds (as in Parseval’s identity) precisely when the family \( \{e_n\} \) is also a “basis,” in the sense that \( \|\sum_{|n|\leq N} a_n e_n - f\| \to 0 \) as \( N \to \infty \).

**Remark 2.** We may associate to every integrable function the sequence \( \{a_n\} \) formed by its Fourier coefficients. Parseval’s identity guarantees that \( \{a_n\} \in \ell^2(\mathbb{Z}) \). Since \( \ell^2(\mathbb{Z}) \) is a Hilbert space, the failure of \( \mathcal{R} \) to be complete, discussed earlier, may be understood as follows: there exist sequences \( \{a_n\}_{n \in \mathbb{Z}} \) such that \( \sum_{n \in \mathbb{Z}} |a_n|^2 < \infty \), yet no Riemann integrable function \( F \) has \( n^{th} \) Fourier coefficient equal to \( a_n \) for all \( n \). An example is given in Exercise 6.

Since the terms of a converging series tend to 0, we deduce from Parseval’s identity or Bessel’s inequality the following result.

**Theorem 1.4 (Riemann-Lebesgue lemma)** If \( f \) is integrable on the circle, then \( \hat{f}(n) \to 0 \) as \( |n| \to \infty \).

An equivalent reformulation of this proposition is that if \( f \) is integrable on \([0, 2\pi]\), then

\[
\int_{0}^{2\pi} f(\theta) \sin(N\theta) \, d\theta \to 0 \quad \text{as} \, N \to \infty.
\]
and

\[ \int_0^{2\pi} f(\theta) \cos(N\theta) \, d\theta \to 0 \quad \text{as } N \to \infty. \]

To conclude this section, we give a more general version of the Parseval identity which we will use in the next chapter.

**Lemma 1.5** Suppose \( F \) and \( G \) are integrable on the circle with

\[ F \sim \sum a_n e^{in\theta} \quad \text{and} \quad G \sim \sum b_n e^{in\theta}. \]

Then

\[ \frac{1}{2\pi} \int_0^{2\pi} F(\theta) \overline{G(\theta)} \, d\theta = \sum_{n=-\infty}^{\infty} a_n \overline{b_n}. \]

Recall from the discussion in Example 1 that the series \( \sum_{n=-\infty}^{\infty} a_n \overline{b_n} \) converges absolutely.

**Proof.** The proof follows from Parseval’s identity and the fact that

\[ (F, G) = \frac{1}{4} \left( \|F + G\|^2 - \|F - G\|^2 + i \left( \|F + iG\|^2 - \|F - iG\|^2 \right) \right) \]

which holds in every Hermitian inner product space. The verification of this fact is left to the reader.

### 2 Return to pointwise convergence

The mean-square convergence theorem does not provide further insight into the problem of pointwise convergence. Indeed, Theorem 1.1 by itself does not guarantee that the Fourier series converges for any \( \theta \). Exercise 3 helps to explain this statement. However, if a function is differentiable at a point \( \theta_0 \), then its Fourier series converges at \( \theta_0 \). After proving this result, we give an example of a continuous function with diverging Fourier series at one point. These phenomena are indicative of the intricate nature of the problem of pointwise convergence in the theory of Fourier series.

#### 2.1 A local result

**Theorem 2.1** Let \( f \) be an integrable function on the circle which is differentiable at a point \( \theta_0 \). Then \( S_N(f)(\theta_0) \to f(\theta_0) \) as \( N \) tends to infinity.
Proof. Define

\[ F(t) = \begin{cases} \frac{f(\theta_0 - t) - f(\theta_0)}{t} & \text{if } t \neq 0 \text{ and } |t| < \pi \\ -f'(\theta_0) & \text{if } t = 0. \end{cases} \]

First, \( F \) is bounded near 0 since \( f \) is differentiable there. Second, for all small \( \delta \) the function \( F \) is integrable on \([-\pi, -\delta] \cup [\delta, \pi]\) because \( f \) has this property and \(|t| > \delta\) there. As a consequence of Proposition 1.4 in the appendix, the function \( F \) is integrable on all of \([-\pi, \pi]\). We know that \( S_N(f)(\theta_0) = (f * D_N)(\theta_0) \), where \( D_N \) is the Dirichlet kernel. Since \( \frac{1}{2\pi} \int D_N = 1 \), we find that

\[
S_N(f)(\theta_0) - f(\theta_0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta_0 - t)D_N(t) \, dt - f(\theta_0)
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} [f(\theta_0 - t) - f(\theta_0)]D_N(t) \, dt
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} F(t)tD_N(t) \, dt.
\]

We recall that

\[
tD_N(t) = \frac{t}{\sin(t/2)} \sin((N + 1/2)t),
\]

where the quotient \( \frac{t}{\sin(t/2)} \) is continuous in the interval \([-\pi, \pi]\). Since we can write

\[
\sin((N + 1/2)t) = \sin(Nt) \cos(t/2) + \cos(Nt) \sin(t/2),
\]

we can apply the Riemann-Lebesgue lemma to the Riemann integrable functions \( F(t)t \cos(t/2)/\sin(t/2) \) and \( F(t)t \) to finish the proof of the theorem.

Observe that the conclusion of the theorem still holds if we only assume that \( f \) satisfies a **Lipschitz condition** at \( \theta_0 \); that is,

\[
|f(\theta) - f(\theta_0)| \leq M|\theta - \theta_0|
\]

for some \( M \geq 0 \) and all \( \theta \). This is the same as saying that \( f \) satisfies a Hölder condition of order \( \alpha = 1 \).

A striking consequence of this theorem is the localization principle of Riemann. This result states that the convergence of \( S_N(f)(\theta_0) \) depends only on the behavior of \( f \) near \( \theta_0 \). This is not clear at first, since forming the Fourier series requires integrating \( f \) over the whole circle.
Theorem 2.2 Suppose $f$ and $g$ are two integrable functions defined on the circle, and for some $\theta_0$ there exists an open interval $I$ containing $\theta_0$ such that

$$f(\theta) = g(\theta) \quad \text{for all } \theta \in I.$$ 

Then $S_N(f)(\theta_0) - S_N(g)(\theta_0) \to 0$ as $N$ tends to infinity.

Proof. The function $f - g$ is 0 in $I$, so it is differentiable at $\theta_0$, and we may apply the previous theorem to conclude the proof.

2.2 A continuous function with diverging Fourier series

We now turn our attention to an example of a continuous periodic function whose Fourier series diverges at a point. Thus, Theorem 2.1 fails if the differentiability assumption is replaced by the weaker assumption of continuity. Our counter-example shows that this hypothesis which had appeared plausible, is in fact false; moreover, its construction also illuminates an important principle of the theory.

The principle that is involved here will be referred to as "symmetry-breaking."\(^1\) The symmetry that we have in mind is the symmetry between the frequencies $e^{in\theta}$ and $e^{-in\theta}$ which appear in the Fourier expansion of a function. For example, the partial sum operator $S_N$ is defined in a way that reflects this symmetry. Also, the Dirichlet, Fejér, and Poisson kernels are symmetric in this sense. When we break the symmetry, that is, when we split the Fourier series $\sum_{n=-\infty}^{\infty} a_n e^{in\theta}$ into the two pieces $\sum_{n \geq 0} a_n e^{in\theta}$ and $\sum_{n < 0} a_n e^{in\theta}$, we introduce new and far-reaching phenomena.

We give a simple example. Start with the sawtooth function $f$ which is odd in $\theta$ and which equals $i(\pi - \theta)$ when $0 < \theta < \pi$. Then, by Exercise 8 in Chapter 2, we know that

$$f(\theta) \sim \sum_{n \neq 0} \frac{e^{in\theta}}{n}. \quad (4)$$

Consider now the result of breaking the symmetry and the resulting series

$$\sum_{n=-\infty}^{n=-1} \frac{e^{in\theta}}{n}.$$ 

Then, unlike (4), the above is no longer the Fourier series of a Riemann integrable function. Indeed, suppose it were the Fourier series of an

---

\(^1\)We have borrowed this terminology from physics, where it is used in a very different context.
integrable function, say $\tilde{f}$, where in particular $\tilde{f}$ is bounded. Using the Abel means, we then have

$$|A_r(\tilde{f})(0)| = \sum_{n=1}^{\infty} \frac{r^n}{n},$$

which tends to infinity as $r$ tends to 1, because $\sum 1/n$ diverges. This gives the desired contradiction since

$$|A_r(\tilde{f})(0)| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |\tilde{f}(\theta)| P_r(\theta) \, d\theta \leq \sup_{\theta} |\tilde{f}(\theta)|,$$

where $P_r(\theta)$ denotes the Poisson kernel discussed in the previous chapter.

The sawtooth function is the object from which we will fashion our counter-example. We proceed as follows. For each $N \geq 1$ we define the following two functions on $[-\pi, \pi]$,

$$f_N(\theta) = \sum_{1 \leq |n| \leq N} \frac{e^{in\theta}}{n} \quad \text{and} \quad \tilde{f}_N(\theta) = \sum_{-N \leq n \leq -1} \frac{e^{in\theta}}{n}.$$

We contend that:

(i) $|\tilde{f}_N(0)| \geq c \log N$.

(ii) $f_N(\theta)$ is uniformly bounded in $N$ and $\theta$.

The first statement is a consequence of the fact that $\sum_{n=1}^{N} 1/n \geq \log N$, which is easily established (see also Figure 2):

$$\sum_{n=1}^{N} \frac{1}{n} \geq \sum_{n=1}^{N-1} \int_{n}^{n+1} \frac{dx}{x} = \int_{1}^{N} \frac{dx}{x} = \log N.$$

To prove (ii), we argue in the same spirit as in the proof of Tauber’s theorem, which says that if the series $\sum c_n$ is Abel summable to $s$ and $c_n = o(1/n)$, then $\sum c_n$ actually converges to $s$ (see Exercise 14 in Chapter 2). In fact, the proof of Tauber’s theorem is quite similar to that of the lemma below.

**Lemma 2.3** Suppose that the Abel means $A_r = \sum_{n=1}^{\infty} r^n c_n$ of the series $\sum_{n=1}^{\infty} c_n$ are bounded as $r$ tends to 1 (with $r < 1$). If $c_n = O(1/n)$, then the partial sums $S_N = \sum_{n=1}^{N} c_n$ are bounded.
2. Return to pointwise convergence

\[ y = \frac{1}{x} \]

\[ n \quad n + 1 \]

**Figure 2.** Comparing a sum with an integral

**Proof.** Let \( r = 1 - 1/N \) and choose \( M \) so that \( n|c_n| \leq M \). We estimate the difference

\[
S_N - A_r = \sum_{n=1}^{N} (c_n - r^n c_n) - \sum_{n=N+1}^{\infty} r^n c_n
\]

as follows:

\[
|S_N - A_r| \leq \sum_{n=1}^{N} |c_n|(1 - r^n) + \sum_{n=N+1}^{\infty} r^n |c_n|
\]

\[
\leq M \sum_{n=1}^{N} (1 - r) + \frac{M}{N} \sum_{n=N+1}^{\infty} r^n
\]

\[
\leq MN(1 - r) + \frac{M}{N} \frac{1}{1 - r}
\]

\[ = 2M, \]

where we have used the simple observation that

\[ 1 - r^n = (1 - r)(1 + r + \cdots + r^{n-1}) \leq n(1 - r). \]

So we see that if \( M \) satisfies both \( |A_r| \leq M \) and \( n|c_n| \leq M \), then \( |S_N| \leq 3M \).

We apply the lemma to the series

\[
\sum_{n \neq 0} \frac{e^{in\theta}}{n},
\]
which is the Fourier series of the sawtooth function $f$ used above. Here $c_n = e^{in\theta}/n + e^{-in\theta}/(-n)$ for $n \neq 0$, so clearly $c_n = O(1/|n|)$. Finally, the Abel means of this series are $A_r(f)(\theta) = (f \ast P_r)(\theta)$. But $f$ is bounded and $P_r$ is a good kernel, so $S_N(f)(\theta)$ is uniformly bounded in $N$ and $\theta$, as was to be shown.

We now come to the heart of the matter. Notice that $f_N$ and $\tilde{f}_N$ are trigonometric polynomials of degree $N$ (that is, they have non-zero Fourier coefficients only when $|n| \leq N$). From these, we form trigonometric polynomials $P_N$ and $\tilde{P}_N$, now of degrees $3N$ and $2N - 1$, by displacing the frequencies of $f_N$ and $\tilde{f}_N$ by $2N$ units. In other words, we define $P_N(\theta) = e^{i(2N)\theta} f_N(\theta)$ and $\tilde{P}_N(\theta) = e^{i(2N)\theta} \tilde{f}_N(\theta)$. So while $f_N$ has non-vanishing Fourier coefficients when $0 < |n| \leq N$, now the coefficients of $P_N$ are non-vanishing for $N \leq n \leq 3N$, $n \neq 2N$. Moreover, while $n = 0$ is the center of symmetry of $f_N$, now $n = 2N$ is the center of symmetry of $P_N$. We next consider the partial sums $S_M$.

**Lemma 2.4**

$$S_M(P_N) = \begin{cases} P_N & \text{if } M \geq 3N, \\ \tilde{P}_N & \text{if } M = 2N, \\ 0 & \text{if } M < N. \end{cases}$$

This is clear from what has been said above and from Figure 3.

![Figure 3. Breaking symmetry in Lemma 2.4](image)

The effect is that when $M = 2N$, the operator $S_M$ breaks the symmetry of $P_N$, but in the other cases covered in the lemma, the action of $S_M$
is relatively benign, since then the outcome is either $P_N$ or 0.

Finally, we need to find a convergent series of positive terms $\sum \alpha_k$ and a sequence of integers $\{N_k\}$ which increases rapidly enough so that:

(i) $N_{k+1} > 3N_k$,

(ii) $\alpha_k \log N_k \to \infty$ as $k \to \infty$.

We choose (for example) $\alpha_k = 1/k^2$ and $N_k = 3^2^k$ which are easily seen to satisfy the above criteria.

Finally, we can write down our desired function. It is

$$f(\theta) = \sum_{k=1}^{\infty} \alpha_k P_{N_k}(\theta).$$

Due to the uniform boundedness of the $P_N$ (recall that $|P_N(\theta)| = |f_N(\theta)|$), the series above converges uniformly to a continuous periodic function. However, by our lemma we get

$$|S_{2N_m}(f)(0)| \geq c\alpha_m \log N_m + O(1) \to \infty \quad \text{as } m \to \infty.$$

Figure 4. Symmetry broken in the middle interval $(N_k, 3N_k)$

Indeed, the terms that correspond to $N_k$ with $k < m$ or $k > m$ contribute $O(1)$ or 0, respectively (because the $P_N$’s are uniformly bounded), while the term that corresponds to $N_m$ is in absolute value greater than $c\alpha_m \log N_m$ because $|\tilde{P}_N(\theta)| = |\tilde{f}_N(\theta)| \geq c \log N$. So the partial sums of the Fourier series of $f$ at 0 are not bounded, and we are done since this proves the divergence of the Fourier series of $f$ at $\theta = 0$. To produce a function whose series diverges at any other preassigned $\theta = \theta_0$, it suffices to consider the function $f(\theta - \theta_0)$.

3 Exercises

1. Show that the first two examples of inner product spaces, namely $\mathbb{R}^d$ and $\mathbb{C}^d$, are complete.
[Hint: Every Cauchy sequence in \( \mathbb{R} \) has a limit.]

2. Prove that the vector space \( \ell^2(\mathbb{Z}) \) is complete.

[Hint: Suppose \( A_k = \{a_{k,n}\}_{n \in \mathbb{Z}} \) with \( k = 1, 2, \ldots \) is a Cauchy sequence. Show that for each \( n \), \( \{a_{k,n}\}_{k=1}^{\infty} \) is a Cauchy sequence of complex numbers, therefore it converges to a limit, say \( b_n \). By taking partial sums of \( \|A_k - A_{k'}\| \) and letting \( k' \to \infty \), show that \( \|A_k - B\| \to 0 \) as \( k \to \infty \), where \( B = (\ldots, b_{-1}, b_0, b_1, \ldots) \). Finally, prove that \( B \in \ell^2(\mathbb{Z}) \).]

3. Construct a sequence of integrable functions \( \{f_k\} \) on \([0, 2\pi]\) such that

\[
\lim_{k \to \infty} \frac{1}{2\pi} \int_0^{2\pi} |f_k(\theta)|^2 \, d\theta = 0
\]

but \( \lim_{k \to \infty} f_k(\theta) \) fails to exist for any \( \theta \).

[Hint: Choose a sequence of intervals \( I_k \subset [0, 2\pi] \) whose lengths tend to 0, and so that each point belongs to infinitely many of them; then let \( f_k = \chi_{I_k} \).]

4. Recall the vector space \( \mathcal{R} \) of integrable functions, with its inner product and norm

\[
\|f\| = \left( \frac{1}{2\pi} \int_0^{2\pi} |f(x)|^2 \, dx \right)^{1/2}.
\]

(a) Show that there exist non-zero integrable functions \( f \) for which \( \|f\| = 0 \).

(b) However, show that if \( f \in \mathcal{R} \) with \( \|f\| = 0 \), then \( f(x) = 0 \) whenever \( f \) is continuous at \( x \).

(c) Conversely, show that if \( f \in \mathcal{R} \) vanishes at all of its points of continuity, then \( \|f\| = 0 \).

5. Let

\[
f(\theta) = \begin{cases} 
0 & \text{for } \theta = 0 \\
\log(1/\theta) & \text{for } 0 < \theta \leq 2\pi,
\end{cases}
\]

and define a sequence of functions in \( \mathcal{R} \) by

\[
f_n(\theta) = \begin{cases} 
0 & \text{for } 0 \leq \theta \leq 1/n \\
f(\theta) & \text{for } 1/n < \theta \leq 2\pi.
\end{cases}
\]

Prove that \( \{f_n\}_{n=1}^{\infty} \) is a Cauchy sequence in \( \mathcal{R} \). However, \( f \) does not belong to \( \mathcal{R} \).
3. Exercises

[Hint: Show that \( \int_a^b (\log \theta)^2 \; d\theta \to 0 \) if \( 0 < a < b \) and \( b \to 0 \), by using the fact that the derivative of \( \theta (\log \theta)^2 - 2 \theta \log \theta + 2\theta \) is equal to \( (\log \theta)^2 \).]

6. Consider the sequence \( \{a_k\}_{k=-\infty}^{\infty} \) defined by

\[
a_k = \begin{cases} 
1/k & \text{if } k \geq 1 \\
0 & \text{if } k \leq 0.
\end{cases}
\]

Note that \( \{a_k\} \in l^2(\mathbb{Z}) \), but that no Riemann integrable function has \( k \)-th Fourier coefficient equal to \( a_k \) for all \( k \).

7. Show that the trigonometric series

\[
\sum_{n \geq 2} \frac{1}{\log n} \sin nx
\]

converges for every \( x \), yet it is not the Fourier series of a Riemann integrable function.

The same is true for \( \sum \frac{\sin nx}{n^{\alpha}} \) for \( 0 < \alpha < 1 \), but the case \( 1/2 < \alpha < 1 \) is more difficult. See Problem 1.

8. Exercise 6 in Chapter 2 dealt with the sums

\[
\sum_{n \; \text{odd} \geq 1} \frac{1}{n^2} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^2}.
\]

Similar sums can be derived using the methods of this chapter.

(a) Let \( f \) be the function defined on \([-\pi, \pi]\) by \( f(\theta) = |\theta| \). Use Parseval's identity to find the sums of the following two series:

\[
\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^4}.
\]

In fact, they are \( \pi^4/96 \) and \( \pi^4/90 \), respectively.

(b) Consider the \( 2\pi \)-periodic odd function defined on \([0, \pi]\) by \( f(\theta) = \theta (\pi - \theta) \). Show that

\[
\sum_{n=0}^{\infty} \frac{1}{(2n+1)^6} = \frac{\pi^6}{960} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}.
\]

Remark. The general expression when \( k \) is even for \( \sum_{n=1}^{\infty} 1/n^k \) in terms of \( \pi^k \) is given in Problem 4. However, finding a formula for the sum \( \sum_{n=1}^{\infty} 1/n^3 \), or more generally \( \sum_{n=1}^{\infty} 1/n^k \) with \( k \) odd, is a famous unresolved question.
9. Show that for $\alpha$ not an integer, the Fourier series of
\[ \frac{\pi}{\sin \pi \alpha} e^{i(\pi - x)\alpha} \]
on [0, 2\pi] is given by
\[ \sum_{n=-\infty}^{\infty} \frac{e^{inx}}{n + \alpha}. \]
Apply Parseval’s formula to show that
\[ \sum_{n=-\infty}^{\infty} \frac{1}{(n + \alpha)^2} = \frac{\pi^2}{(\sin \pi \alpha)^2}. \]

10. Consider the example of a vibrating string which we analyzed in Chapter 1. The displacement $u(x, t)$ of the string at time $t$ satisfies the wave equation
\[ \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad c^2 = \tau / \rho. \]
The string is subject to the initial conditions
\[ u(x, 0) = f(x) \quad \text{and} \quad \frac{\partial u}{\partial t}(x, 0) = g(x), \]
where we assume that $f \in C^1$ and $g$ is continuous. We define the total energy of the string by
\[ E(t) = \frac{1}{2} \rho \int_0^L \left( \frac{\partial u}{\partial t} \right)^2 dx + \frac{1}{2} T \int_0^L \left( \frac{\partial u}{\partial x} \right)^2 dx. \]
The first term corresponds to the “kinetic energy” of the string (in analogy with $(1/2)m v^2$, the kinetic energy of a particle of mass $m$ and velocity $v$), and the second term corresponds to its “potential energy.”
Show that the total energy of the string is conserved, in the sense that $E(t)$ is constant. Therefore,
\[ E(t) = E(0) = \frac{1}{2} \rho \int_0^L g(x)^2 dx + \frac{1}{2} T \int_0^L f'(x)^2 dx. \]

11. The inequalities of Wirtinger and Poincaré establish a relationship between the norm of a function and that of its derivative.
(a) If \( f \) is \( T \)-periodic, continuous, and piecewise \( C^1 \) with \( \int_{0}^{T} f(t) \, dt = 0 \), show that
\[
\int_{0}^{T} |f(t)|^2 \, dt \leq \frac{T^2}{4\pi^2} \int_{0}^{T} |f'(t)|^2 \, dt,
\]
with equality if and only if \( f(t) = A\sin(2\pi t/T) + B\cos(2\pi t/T) \). [Hint: Apply Parseval's identity.]

(b) If \( f \) is as above and \( g \) is just \( C^1 \) and \( T \)-periodic, prove that
\[
\left| \int_{0}^{T} f(t)g(t) \, dt \right|^2 \leq \frac{T^2}{4\pi^2} \int_{0}^{T} |f(t)|^2 \, dt \int_{0}^{T} |g'(t)|^2 \, dt.
\]

(c) For any compact interval \([a, b]\) and any continuously differentiable function \( f \) with \( f(a) = f(b) = 0 \), show that
\[
\int_{a}^{b} |f(t)|^2 \, dt \leq \frac{(b - a)^2}{\pi^2} \int_{a}^{b} |f'(t)|^2 \, dt.
\]
Discuss the case of equality, and prove that the constant \( (b - a)^2/\pi^2 \) cannot be improved. [Hint: Extend \( f \) to be odd with respect to \( a \) and periodic of period \( T = 2(b - a) \) so that its integral over an interval of length \( T \) is 0. Apply part a) to get the inequality, and conclude that equality holds if and only if \( f(t) = A\sin(\pi \frac{t-a}{b-a}) \).]

12. Prove that \( \int_{0}^{\infty} \frac{\sin x}{x} \, dx = \frac{\pi}{2} \).
[Hint: Start with the fact that the integral of \( D_N(\theta) \) equals \( 2\pi \), and note that the difference \( (1/\sin(\theta/2)) - 2/\theta \) is continuous on \([-\pi, \pi]\). Apply the Riemann-Lebesgue lemma.]

13. Suppose that \( f \) is periodic and of class \( C^k \). Show that
\[
\hat{f}(n) = o(1/|n|^k),
\]
that is, \( |n|^k \hat{f}(n) \) goes to 0 as \( |n| \to \infty \). This is an improvement over Exercise 10 in Chapter 2.
[Hint: Use the Riemann-Lebesgue lemma.]

14. Prove that the Fourier series of a continuously differentiable function \( f \) on the circle is absolutely convergent.
[Hint: Use the Cauchy-Schwarz inequality and Parseval's identity for \( f' \).]

15. Let \( f \) be \( 2\pi \)-periodic and Riemann integrable on \([-\pi, \pi]\).
(a) Show that

\[ \hat{f}(n) = -\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x + \pi/n)e^{-inx} \, dx \]

hence

\[ \hat{f}(n) = \frac{1}{4\pi} \int_{-\pi}^{\pi} [f(x) - f(x + \pi/n)]e^{-inx} \, dx. \]

(b) Now assume that \( f \) satisfies a Hölder condition of order \( \alpha \), namely

\[ |f(x + h) - f(x)| \leq C|h|^{\alpha} \]

for some \( 0 < \alpha \leq 1 \), some \( C > 0 \), and all \( x, h \). Use part a) to show that

\[ \hat{f}(n) = O(1/|n|^{\alpha}). \]

(c) Prove that the above result cannot be improved by showing that the function

\[ f(x) = \sum_{k=0}^{\infty} 2^{-k\alpha} e^{i2^k x}, \]

where \( 0 < \alpha < 1 \), satisfies

\[ |f(x + h) - f(x)| \leq C|h|^{\alpha}, \]

and \( \hat{f}(N) = 1/N^\alpha \) whenever \( N = 2^k \).

**Hint:** For (c), break up the sum as follows \( f(x + h) - f(x) = \sum_{2^k \leq 1/|h|} + \sum_{2^k > 1/|h|} \). To estimate the first sum use the fact that \( |1 - e^{i\theta}| \leq |\theta| \) whenever \( \theta \) is small. To estimate the second sum, use the obvious inequality \( |e^{ix} - e^{iy}| \leq 2 \).

16. Let \( f \) be a \( 2\pi \)-periodic function which satisfies a Lipschitz condition with constant \( K \); that is,

\[ |f(x) - f(y)| \leq K|x - y| \quad \text{for all } x, y. \]

This is simply the Hölder condition with \( \alpha = 1 \), so by the previous exercise, we see that \( \hat{f}(n) = O(1/|n|) \). Since the harmonic series \( \sum 1/n \) diverges, we cannot say anything (yet) about the absolute convergence of the Fourier series of \( f \). The outline below actually proves that the Fourier series of \( f \) converges absolutely and uniformly.
3. Exercises

(a) For every positive \( h \) we define \( g_h(x) = f(x + h) - f(x - h) \). Prove that

\[
\frac{1}{2\pi} \int_0^{2\pi} |g_h(x)|^2 \, dx = \sum_{n=-\infty}^{\infty} 4|\sin nh|^2 |\hat{f}(n)|^2,
\]

and show that

\[
\sum_{n=-\infty}^{\infty} |\sin nh|^2 |\hat{f}(n)|^2 \leq K^2 h^2.
\]

(b) Let \( p \) be a positive integer. By choosing \( h = \pi/2^{p+1} \), show that

\[
\sum_{2^{p-1} < |n| \leq 2^p} |\hat{f}(n)|^2 \leq \frac{K^2 \pi^2}{2^{2p+1}}.
\]

(c) Estimate \( \sum_{2^{p-1} < |n| \leq 2^p} |\hat{f}(n)| \), and conclude that the Fourier series of \( f \) converges absolutely, hence uniformly. [Hint: Use the Cauchy-Schwarz inequality to estimate the sum.]

(d) In fact, modify the argument slightly to prove Bernstein’s theorem: If \( f \) satisfies a Hölder condition of order \( \alpha > 1/2 \), then the Fourier series of \( f \) converges absolutely.

17. If \( f \) is a bounded monotonic function on \([-\pi, \pi]\), then

\[ \hat{f}(n) = O(1/|n|). \]

[Hint: One may assume that \( f \) is increasing, and say \(|f| \leq M \). First check that the Fourier coefficients of the characteristic function of \([a, b]\) satisfy \( O(1/|n|) \). Now show that a sum of the form

\[ \sum_{k=1}^{N} \alpha_k \chi_{[a_k, a_{k+1}]}(x) \]

with \(-\pi = a_1 < a_2 < \cdots < a_N < a_{N+1} = \pi \) and \(-M \leq \alpha_1 \leq \cdots \leq \alpha_N \leq M \) has Fourier coefficients that are \( O(1/|n|) \) uniformly in \( N \). Summing by parts one gets a telescopic sum \( \sum (\alpha_{k+1} - \alpha_k) \) which can be bounded by \( 2M \). Now approximate \( f \) by functions of the above type.]

18. Here are a few things we have learned about the decay of Fourier coefficients:

(a) if \( f \) is of class \( C^k \), then \( \hat{f}(n) = o(1/|n|^k) \);

(b) if \( f \) is Lipschitz, then \( \hat{f}(n) = O(1/|n|) \);
(c) if $f$ is monotonic, then $\hat{f}(n) = O(1/|n|)$;

(d) if $f$ is satisfies a Hölder condition with exponent $\alpha$ where $0 < \alpha < 1$, then $\hat{f}(n) = O(1/|n|^\alpha)$;

(e) if $f$ is merely Riemann integrable, then $\sum |\hat{f}(n)|^2 < \infty$ and therefore $\hat{f}(n) = o(1)$.

Nevertheless, show that the Fourier coefficients of a continuous function can tend to 0 arbitrarily slowly by proving that for every sequence of nonnegative real numbers $\{\epsilon_n\}$ converging to 0, there exists a continuous function $f$ such that $|\hat{f}(n)| \geq \epsilon_n$ for infinitely many values of $n$.

[Hint: Choose a subsequence $\{\epsilon_{n_k}\}$ so that $\sum_k \epsilon_{n_k} < \infty$.]

19. Give another proof that the sum $\sum_{0 < |n| \leq N} e^{inx}/n$ is uniformly bounded in $N$ and $x \in [-\pi, \pi]$ by using the fact that

$$\frac{1}{2i} \sum_{0 < |n| \leq N} \frac{e^{inx}}{n} = \sum_{n=1}^{N} \frac{\sin nx}{n} = \frac{1}{2} \int_{0}^{x} (D_N(t) - 1) \, dt,$$

where $D_N$ is the Dirichlet kernel. Now use the fact that $\int_{0}^{\infty} \frac{\sin t}{t} \, dt < \infty$ which was proved in Exercise 12.

20. Let $f(x)$ denote the sawtooth function defined by $f(x) = (\pi - x)/2$ on the interval $(0, 2\pi)$ with $f(0) = 0$ and extended by periodicity to all of $\mathbb{R}$. The Fourier series of $f$ is

$$f(x) \sim \frac{1}{2i} \sum_{|n| \neq 0} \frac{e^{inx}}{n} = \sum_{n=1}^{\infty} \frac{\sin nx}{n},$$

and $f$ has a jump discontinuity at the origin with

$$f(0^+) = \frac{\pi}{2}, \quad f(0^-) = -\frac{\pi}{2}, \quad \text{and hence} \quad f(0^+) - f(0^-) = \pi.$$

Show that

$$\max_{0 < x \leq \pi/N} S_N(f)(x) - \frac{\pi}{2} = \int_{0}^{\pi} \frac{\sin t}{t} \, dt - \frac{\pi}{2},$$

which is roughly 9% of the jump $\pi$. This result is a manifestation of Gibbs's phenomenon which states that near a jump discontinuity, the Fourier series of a function overshoots (or undershoots) it by approximately 9% of the jump.

[Hint: Use the expression for $S_N(f)$ given in Exercise 19.]
4 Problems

1. For each $0 < \alpha < 1$ the series
\[ \sum_{n=1}^{\infty} \frac{\sin nx}{n^\alpha} \]
converges for every $x$ but is not the Fourier series of a Riemann integrable function.

(a) If the conjugate Dirichlet kernel is defined by
\[ \tilde{D}_N(x) = \sum_{|n|\leq N} \text{sign}(x) e^{inx} \quad \text{where} \quad \text{sign}(x) = \begin{cases} 1 & \text{if } n > 0 \\ 0 & \text{if } n = 0 \\ -1 & \text{if } n < 0, \end{cases} \]
en then show that
\[ \tilde{D}_N(x) = \frac{\cos(x/2) - \cos((N + 1/2)x)}{\sin(x/2)}, \]
and
\[ \int_{-\pi}^{\pi} |\tilde{D}_N(x)| \, dx \leq c \log N. \]

(b) As a result, if $f$ is Riemann integrable, then
\[ (f * \tilde{D}_N)(0) = O(\log N). \]

(c) In the present case, this leads to
\[ \sum_{n=1}^{N} \frac{1}{n^\alpha} = O(\log N), \]
which is a contradiction.

2. An important fact we have proved is that the family $\{e^{inx}\}_{n\in\mathbb{Z}}$ is orthonormal in $\mathcal{R}$ and it is also complete, in the sense that the Fourier series of $f$ converges to $f$ in the norm. In this exercise, we consider another family possessing these same properties.

On $[-1, 1]$ define
\[ L_n(x) = \frac{d^n}{dx^n} (x^2 - 1)^n, \quad n = 0, 1, 2, \ldots. \]
Then $L_n$ is a polynomial of degree $n$ which is called the $n^{th}$ Legendre polynomial.
(a) Show that if \( f \) is indefinitely differentiable on \([-1, 1]\), then
\[
\int_{-1}^{1} L_n(x) f(x) \, dx = (-1)^n \int_{-1}^{1} (x^2 - 1)^n f^{(n)}(x) \, dx.
\]

In particular, show that \( L_n \) is orthogonal to \( x^m \) whenever \( m < n \). Hence \( \{L_n\}_{n=0}^\infty \) is an orthogonal family.

(b) Show that
\[
\|L_n\|^2 = \int_{-1}^{1} |L_n(x)|^2 \, dx = \frac{(n!)^2 2^{2n+1}}{2n + 1}.
\]

[Hint: First, note that \( \|L_n\|^2 = (-1)^n (2n)! \int_{-1}^{1} (x^2 - 1)^n \, dx \). Write \( (x^2 - 1)^n = (x - 1)^n (x + 1)^n \) and integrate by parts \( n \) times to calculate this last integral.]

(c) Prove that any polynomial of degree \( n \) that is orthogonal to \( 1, x, x^2, \ldots, x^{n-1} \) is a constant multiple of \( L_n \).

(d) Let \( L_n = L_n/\|L_n\| \), which are the normalized Legendre polynomials. Prove that \( \{L_n\} \) is the family obtained by applying the “Gram-Schmidt process” to \( \{1, x, \ldots, x^n, \ldots\} \), and conclude that every Riemann integrable function \( f \) on \([-1, 1]\) has a Legendre expansion
\[
\sum_{n=0}^{\infty} \langle f, L_n \rangle L_n
\]
which converges to \( f \) in the mean-square sense.

3. Let \( \alpha \) be a complex number not equal to an integer.

(a) Calculate the Fourier series of the \( 2\pi \)-periodic function defined on \([-\pi, \pi]\) by \( f(x) = \cos(\alpha x) \).

(b) Prove the following formulas due to Euler:
\[
\sum_{n=1}^{\infty} \frac{1}{n^2 - \alpha^2} = \frac{1}{2\alpha^2} - \frac{\pi}{2\alpha \tan(\alpha \pi)}.
\]

For all \( u \in \mathbb{C} - \pi \mathbb{Z} \),
\[
\cot u = \frac{1}{u} + 2 \sum_{n=1}^{\infty} \frac{u}{u^2 - n^2 \pi^2}.
\]
(c) Show that for all $\alpha \in \mathbb{C} - \mathbb{Z}$ we have

$$\frac{\alpha \pi}{\sin(\alpha \pi)} = 1 + 2\alpha^2 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2 - \alpha^2}.$$

(d) For all $0 < \alpha < 1$, show that

$$\int_{0}^{\infty} \frac{t^{\alpha-1}}{t+1} \, dt = \frac{\pi}{\sin(\alpha \pi)}.$$

[Hint: Split the integral as $\int_{0}^{1} + \int_{1}^{\infty}$ and change variables $t = 1/u$ in the second integral. Now both integrals are of the form

$$\int_{0}^{1} \frac{t^{\gamma-1}}{1+t} \, dt, \quad 0 < \gamma < 1,$$

which one can show is equal to $\sum_{k=0}^{\infty} \frac{(-1)^k}{k+\gamma}$. Use part (c) to conclude the proof.]

4. In this problem, we find the formula for the sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{n^k}$$

where $k$ is any even integer. These sums are expressed in terms of the Bernoulli numbers; the related Bernoulli polynomials are discussed in the next problem.

Define the Bernoulli numbers $B_n$ by the formula

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n.$$

(a) Show that $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_3 = 0$, $B_4 = -1/30$, and $B_5 = 0$.

(b) Show that for $n \geq 1$ we have

$$B_n = -\frac{1}{n+1} \sum_{k=0}^{n-1} \binom{n+1}{k} B_k.$$

(c) By writing

$$\frac{z}{e^z - 1} = 1 - \frac{z}{2} + \sum_{n=2}^{\infty} \frac{B_n}{n!} z^n,$$
show that $B_n = 0$ if $n$ is odd and $> 1$. Also prove that

$$z \cot z = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{2^{2n}B_{2n}}{(2n)!} z^{2n}.$$  

(d) The **zeta function** is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \text{for all } s > 1.$$  

Deduce from the result in (c), and the expression for the cotangent function obtained in the previous problem, that

$$x \cot x = 1 - 2 \sum_{m=1}^{\infty} \frac{\zeta(2m)}{\pi^{2m}} x^{2m}.$$  

(e) Conclude that

$$2\zeta(2m) = (-1)^{m+1} \frac{(2\pi)^{2m}}{(2m)!} B_{2m}.$$  

5. Define the **Bernoulli polynomials** $B_n(x)$ by the formula

$$\frac{ze^{zx}}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} z^n.$$  

(a) The functions $B_n(x)$ are polynomials in $x$ and

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}.$$  

Show that $B_0(x) = 1, \ B_1(x) = x - 1/2, \ B_2(x) = x^2 - x + 1/6, \text{ and } B_3(x) = x^3 - \frac{3}{2} x^2 + \frac{1}{2} x.$

(b) If $n \geq 1$, then

$$B_n(x+1) - B_n(x) = nx^{n-1},$$

and if $n \geq 2$, then

$$B_n(0) = B_n(1) = B_n.$$  

(c) Define $S_m(n) = 1^m + 2^m + \cdots + (n-1)^m$. Show that

$$(m+1)S_m(n) = B_{m+1}(n) - B_{m+1}.$$
4. Problems

(d) Prove that the Bernoulli polynomials are the only polynomials that satisfy

(i) \( B_0(x) = 1, \)
(ii) \( B'_n(x) = nB_{n-1}(x) \) for \( n \geq 1, \)
(iii) \( \int_0^1 B_n(x) \, dx = 0 \) for \( n \geq 1, \) and show that from (b) one obtains

\[
\int_x^{x+1} B_n(t) \, dt = x^n.
\]

(e) Calculate the Fourier series of \( B_1(x) \) to conclude that for \( 0 < x < 1 \) we have

\[
B_1(x) = x - 1/2 = -\frac{1}{\pi} \sum_{k=1}^{\infty} \frac{\sin(2\pi kx)}{k}.
\]

Integrate and conclude that

\[
B_{2n}(x) = (-1)^{n+1} \frac{2(2n)!}{(2\pi)^{2n}} \sum_{k=1}^{\infty} \frac{\cos(2\pi kx)}{k^{2n}},
\]

\[
B_{2n+1}(x) = (-1)^{n+1} \frac{2(2n + 1)!}{(2\pi)^{2n+1}} \sum_{k=1}^{\infty} \frac{\sin(2\pi kx)}{k^{2n+1}}.
\]

Finally, show that for \( 0 < x < 1, \)

\[
B_n(x) = -\frac{n!}{(2\pi i)^n} \sum_{k \neq 0} \frac{e^{2\pi i kx}}{k^n}.
\]

We observe that the Bernoulli polynomials are, up to normalization, successive integrals of the sawtooth function.