Billiards and curves in moduli space

Curtis T McMullen
Harvard University

Avila, Hubert, Kenyon, Kontsevich, Lanneau, Masur, Smillie, Yoccoz, Zorich, ...

A Little History

\[ \int \frac{dx}{\sqrt{1 - x^2}} = \sin^{-1} x \]

\[ Q(x) \text{ a polynomial} \]

\[ \int \frac{dx}{\sqrt[3]{1 - x^3}} = 3\sqrt[3]{\frac{x-1}{1+\sqrt[3]{x}} + 1} \sqrt[3]{\frac{x-1}{1-\sqrt[3]{x}}} + 1(x-1)F_1 \left( \frac{2}{3}; \frac{1}{3}; \frac{1}{3}; -\frac{x-1}{1-\sqrt[3]{x}}, -\frac{x-1}{1+\sqrt[3]{x}} \right) \]

\[ F_1 = \text{Appell hypergeometric function} \]

\[ M_g = \text{moduli space of Riemann surfaces } X \text{ of genus } g \]

\[ \{ \} \]

-- a complex variety, dimension 3g-3

Teichmüller metric: every holomorphic map

\[ f : \mathbb{H}^2 \to M_g \]

is distance-decreasing.

TOTALLY unsymmetric

→ Riemann surfaces, homology, Hodge theory, automorphic forms, ...
How to describe $X$ in $\mathcal{M}_g$?

$g=1$: $X = \mathbb{C}/\Lambda$

$g>1$: $X = \text{?} \quad \text{Uniformization Theorem}$

Every $X$ in $\mathcal{M}_g$ can be built from a polygon in $\mathbb{C}$

$X = P / \text{gluing by translations}$

---

Moduli space $\Omega\mathcal{M}_g$

**Dynamical:**

$\text{SL}_2(\mathbb{R})$ acts on $\Omega\mathcal{M}_g$

Polygon for $A \cdot (X,\omega) = A \cdot (\text{Polygon for } (X,\omega))$

*Complex geodesics* $f : \mathcal{H} \to \mathcal{M}_g$

---

Teichmüller curves

$\text{SL}_2(\mathbb{R})$ orbit of $(X,\omega)$ in $\Omega\mathcal{M}_g$ projects to a *complex geodesic* in $\mathcal{M}_g$:

$$\mathcal{H} \to \mathcal{M}_g$$

$V = \mathcal{H} / \text{SL}(X,\omega)$

$\text{stabilizer of } (X,\omega)$

$\text{SL}(X,\omega)$ lattice $\leftrightarrow f : V \to \mathcal{M}_g$ is an algebraic, isometrically immersed *Teichmüller curve*.
Rigidity Conjecture

The closure of any complex geodesic \( f(H) \subset \mathcal{M}_g \)

is an algebraic subvariety.

Celebrated theorem of Ratner (1995) \( \Rightarrow \)

true for \( \mathbb{H} \rightarrow \) locally symmetric spaces

\( X = K\backslash G/\Gamma \)

Complex geodesics in genus two

Theorem

Let \( f : \mathbb{H} \rightarrow \mathcal{M}_2 \) be a complex geodesic.

Then \( f(H) \) is either:

- A Teichmüller curve, \( \dim = 1 \)
- A Hilbert modular surface \( H_D \), or \( \dim = 2 \)
- The whole space \( \mathcal{M}_2 \), \( \dim = 3 \)

Recent progress towards general \( g \)

Eskin -- Mirzakhani

Classification Problem

What are the Teichmüller curves \( V \rightarrow \mathcal{M}_2 \)?

Billiards in polygons

Neither periodic nor evenly distributed
Billiard theorists

Optimal Billiards

Theorem. In a regular n-gon, every billiard path is either periodic or uniformly distributed. (Veech)

Billiards and Riemann surfaces

P is a Lattice Polygon

\[ (X, \omega) = P/\sim \]

\( X \) has genus 2
\( \omega \) has just one zero!

\( \Leftrightarrow \) \( SL(X, \omega) \) is a lattice
\( \Leftrightarrow (X, \omega) \) generates a Teichmüller curve

**Theorem** (Veech, Masur): If \( P \) is a lattice polygon, then billiards in \( P \) is optimal. (renormalization)
Optimal Billiards

**Example:** if $X = \mathbb{C}/\Lambda$, $\omega=dz$, then $SL(X,\omega) = SL_2(\mathbb{Z})$

**Theorem (Veech, 1989):** For $(X,\omega) = (y^2 = x^n-1, dx/y)$, $SL(X,\omega)$ is a lattice.

**Corollary**

*Any regular polygon is a lattice polygon.*

---

Explicit package: Pentagon example

$(X,\omega) = (y^2=x^5-1, dx/y)$

$\Rightarrow$ **Direct proof that $SL(X,\omega)$ is a lattice**

---

**20th century lattice billiards**

- **Square** $\sim SL_2(\mathbb{Z})$
- **Tiled by squares** $\sim SL_2(\mathbb{Z})$
- **Regular polygons** $\sim (2,n,\infty)$ triangle group
- **Various triangles** triangle groups

---

Genus 2

$\sim\Rightarrow$ Regular 5- 8- and 10-gon

**Problem**

Are there infinitely many primitive Teichmüller curves $V$ in the moduli space $M_2$?
Jacobians with real multiplication

**Theorem**

\((X, \omega)\) generates a Teichmüller curve \(V\Rightarrow\)

\(\text{Jac}(X)\) admits real multiplication by \(\mathcal{O}_D \subset \mathbb{Q}(\sqrt{D})\).

**Corollary**

\(V\) lies on a Hilbert modular surface

\[ V \subset H_D \subset \mathcal{M}_2 \]

\[ \text{H} \times \text{H} / \text{SL}_2(\mathcal{O}_D) \]

---

The Weierstrass curves

\(W_D = \{X \in \mathcal{M}_2 : \mathcal{O}_D \text{ acts on } \text{Jac}(X) \text{ and its eigenform } \omega \text{ has a double zero.}\}\)

**Theorem.** \(W_D\) is a finite union of Teichmüller curves.

\[ W_D = \{X \in \mathcal{M}_2 : \mathcal{O}_D \text{ acts on } \text{Jac}(X) \text{ and its eigenform } \omega \text{ has a double zero.}\} \]

**Corollaries**

- \(P_d\) has optimal billiards for all integers \(d > 0\).
- There are infinitely many primitive \(V\) in genus 2.

---

The regular decagon

**Theorem.** The only other primitive Teichmüller curve in genus two is generated by the regular decagon.

---

Torsion divisors in genus two

**Theorem (Möller)** \((X, \omega)\) generates a Teichmüller curve \(\Rightarrow [P-Q] \text{ is torsion in } \text{Jac}(X)\)

Teichmüller curves in genus 2

Theorem

The Weierstrass curves $W_D$ account for all the primitive Teichmüller curves in genus 2 --

-- except for the curve coming from the regular decagon.

Mysteries

• Is $W_D$ irreducible?
• What is its Euler characteristic?
• What is its genus?
• Algebraic points $(X, \omega)$ in $W_D$?
• What is $\Gamma = SL(X, \omega)$?

$W_D = \mathbb{H}/\Gamma$, $\Gamma \subset SL_2(\mathcal{O}_D)$

Classification Theorem

$W_D$ is connected except when $D = 1 \mod 8, D > 9$.

Euler characteristic of $W_D$

Theorem (Bainbridge, 2006)

$$\chi(W_D) = -\frac{9}{2}\chi(SL_2(\mathcal{O}_D))$$

= coefficients of a modular form

Compare: $\chi(M_{g,1}) = \zeta(1-2g)$ (Harer-Zagier)

Proof: Uses cusp form on Hilbert modular surface with $(\alpha) = W_D - P_D$, where $P_D$ is a Shimura curve
Elliptic points on $W_D$

*Theorem (Mukamel, 2011)*

The number of orbifold points on $W_D$ is given by a sum of class numbers for $Q(\sqrt{-D})$.

**Proof:** $(X, \omega)$ corresponds to an orbifold point $\Rightarrow$ $X$ covers a CM elliptic curve $E$ $\Rightarrow$ $(X, \omega), p: X \to E$ and $\text{Jac}(X)$ can be described explicitly.

Algebraic points on $W_D$

$X \in M_2$

- $D=5 \quad y^2 = x^5 - 1$
- $D=8 \quad y^2 = x^8 - 1$
- $D=13 \quad y^2 = (x^2 - 1)(x^4 - ax^2 + 1)$
  
  \[ a = 2594 + 720 \sqrt{13} \]
  ....

- $D=108 \quad 96001 + 48003 a + 3 a^2 + a^3 = 0$

Genus of $W_D$

<table>
<thead>
<tr>
<th>$D$</th>
<th>$g(W_D)$</th>
<th>$c_2(W_D)$</th>
<th>$C(W_D)$</th>
<th>$\chi(W_D)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>-1/2</td>
</tr>
<tr>
<td>8</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>-1/8</td>
</tr>
<tr>
<td>9</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>-1/8</td>
</tr>
<tr>
<td>12</td>
<td>0</td>
<td>1</td>
<td>3</td>
<td>-1/8</td>
</tr>
<tr>
<td>13</td>
<td>0</td>
<td>1</td>
<td>3</td>
<td>-1/8</td>
</tr>
<tr>
<td>16</td>
<td>0</td>
<td>1</td>
<td>3</td>
<td>-1/8</td>
</tr>
<tr>
<td>17</td>
<td>0.0</td>
<td>1.1</td>
<td>3.3</td>
<td>-1/8</td>
</tr>
<tr>
<td>20</td>
<td>0</td>
<td>2</td>
<td>4</td>
<td>-3</td>
</tr>
<tr>
<td>21</td>
<td>0</td>
<td>3</td>
<td>5</td>
<td>-3</td>
</tr>
<tr>
<td>24</td>
<td>0</td>
<td>1</td>
<td>6</td>
<td>-3</td>
</tr>
<tr>
<td>25</td>
<td>0.0</td>
<td>0.1</td>
<td>5.3</td>
<td>-3</td>
</tr>
<tr>
<td>28</td>
<td>0</td>
<td>2</td>
<td>7</td>
<td>-6</td>
</tr>
<tr>
<td>29</td>
<td>0</td>
<td>3</td>
<td>5</td>
<td>-2</td>
</tr>
<tr>
<td>32</td>
<td>0</td>
<td>2</td>
<td>7</td>
<td>-6</td>
</tr>
<tr>
<td>33</td>
<td>0.0</td>
<td>1.1</td>
<td>6.6</td>
<td>-3</td>
</tr>
<tr>
<td>36</td>
<td>0</td>
<td>0</td>
<td>8</td>
<td>-6</td>
</tr>
<tr>
<td>37</td>
<td>0</td>
<td>1</td>
<td>9</td>
<td>-1/3</td>
</tr>
<tr>
<td>40</td>
<td>0</td>
<td>1</td>
<td>12</td>
<td>-1/3</td>
</tr>
<tr>
<td>41</td>
<td>0.0</td>
<td>2.2</td>
<td>7.7</td>
<td>-6</td>
</tr>
<tr>
<td>44</td>
<td>1</td>
<td>3</td>
<td>9</td>
<td>-1/2</td>
</tr>
<tr>
<td>45</td>
<td>1</td>
<td>2</td>
<td>8</td>
<td>-9</td>
</tr>
<tr>
<td>49</td>
<td>0.0</td>
<td>2.0</td>
<td>10.8</td>
<td>-9</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$D$</th>
<th>$g(W_D)$</th>
<th>$c_2(W_D)$</th>
<th>$C(W_D)$</th>
<th>$\chi(W_D)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>52</td>
<td>1</td>
<td>0</td>
<td>15</td>
<td>-15</td>
</tr>
<tr>
<td>53</td>
<td>2</td>
<td>3</td>
<td>7</td>
<td>-15</td>
</tr>
<tr>
<td>56</td>
<td>3</td>
<td>2</td>
<td>10</td>
<td>-15</td>
</tr>
<tr>
<td>57</td>
<td>1.1</td>
<td>1.1</td>
<td>10.10</td>
<td>-15</td>
</tr>
<tr>
<td>60</td>
<td>3</td>
<td>4</td>
<td>12</td>
<td>-18</td>
</tr>
<tr>
<td>61</td>
<td>2</td>
<td>3</td>
<td>13</td>
<td>-15</td>
</tr>
<tr>
<td>64</td>
<td>1</td>
<td>2</td>
<td>17</td>
<td>-18</td>
</tr>
<tr>
<td>65</td>
<td>1.1</td>
<td>2.2</td>
<td>11.11</td>
<td>-12</td>
</tr>
<tr>
<td>68</td>
<td>3</td>
<td>0</td>
<td>14</td>
<td>-18</td>
</tr>
<tr>
<td>69</td>
<td>4</td>
<td>4</td>
<td>10</td>
<td>-18</td>
</tr>
<tr>
<td>72</td>
<td>1</td>
<td>6</td>
<td>2</td>
<td>-15</td>
</tr>
<tr>
<td>73</td>
<td>1.1</td>
<td>1.1</td>
<td>16.16</td>
<td>-15</td>
</tr>
<tr>
<td>76</td>
<td>4</td>
<td>3</td>
<td>21</td>
<td>-15</td>
</tr>
<tr>
<td>77</td>
<td>5</td>
<td>4</td>
<td>8</td>
<td>-18</td>
</tr>
<tr>
<td>80</td>
<td>4</td>
<td>4</td>
<td>16</td>
<td>-24</td>
</tr>
<tr>
<td>81</td>
<td>2.0</td>
<td>0.3</td>
<td>16.14</td>
<td>-18</td>
</tr>
<tr>
<td>84</td>
<td>7</td>
<td>0</td>
<td>18</td>
<td>-30</td>
</tr>
<tr>
<td>85</td>
<td>6</td>
<td>2</td>
<td>16</td>
<td>-27</td>
</tr>
<tr>
<td>88</td>
<td>7</td>
<td>1</td>
<td>22</td>
<td>-27</td>
</tr>
<tr>
<td>89</td>
<td>3.3</td>
<td>3.3</td>
<td>14.14</td>
<td>-27</td>
</tr>
<tr>
<td>92</td>
<td>8</td>
<td>6</td>
<td>13</td>
<td>-30</td>
</tr>
<tr>
<td>93</td>
<td>8</td>
<td>2</td>
<td>12</td>
<td>-27</td>
</tr>
<tr>
<td>96</td>
<td>8</td>
<td>4</td>
<td>20</td>
<td>-36</td>
</tr>
</tbody>
</table>

**Corollary**

$W_D$ has genus 0 only for $D < 50$

(table by Mukamel)

Computing $W_D$

$D=44$

Mukamel
**Conjecture:**
There are only finitely many Teichmüller curves in $\mathcal{M}_g$ with $\deg(\text{trace field } SL(X,\omega)) = g = 3$ or more.

(avoid echo of lower genera)

**[Rules out quadratic fields]**

---

**Theorem (Möller, Bainbridge-Möller):**
Finiteness holds...
for hyperelliptic stratum $(g-1,g-1)$
for $g=3$, stratum $(3,1)$

**Methods:** Variation of Hodge structure; rigidity theorems of Deligne and Schmid; Neron models; arithmetic geometry

- Jac$(X)$ admits real multiplication by $K$,
- $P-Q$ is torsion in Jac$(X)$ for any two zeros of $\omega$.

---

**However...**

**Theorem**
There exist infinitely many primitive Teichmüller curves in $\mathcal{M}_g$ for genus $g = 2, 3$ and $4$.

---

**Exceptional triangular billiards**

- $\frac{1}{3}$
- $\frac{1}{5}$
- $\frac{7}{15}$
- $\frac{4}{9}$
- $\frac{2}{9}$
- $\frac{5}{12}$
- $\frac{1}{4}$
Prym systems in genus 2, 3 and 4

Higher genus?

Question.
Are there only finitely many primitive Teichmüller curves in \( \mathcal{M}_g \) for each \( g \geq 5 \)?

What about the Hilbert modular surfaces \( H_D \subset \mathcal{M}_2 \)?

\begin{align*}
\mathbb{H} \times \mathbb{H} & \hspace{1cm} \text{foliated by complex geodesics} \\
\downarrow & \\
H_D \subset \mathcal{M}_2 & \\
\end{align*}

each leaf is the graph of a holomorphic function \( F: \mathbb{H} \rightarrow \mathbb{H} \)

W_\mathcal{D} for \( g=3,4 \): Lanneau--Nguyen but still quadratic fields

Pentagon-to-star map

\( \tilde{W}_5 = \text{graph of } F \)
**Action on slices of $H_D$**

Slice $\{\tau_1\} \times \mathbb{H}$

$\rho = \int_a^b \omega = $ relative period

$q = (d\rho)^2$ quadratic differential

$SL(\mathbb{H}, q) = SL_2(\mathbb{D})$

*acts on slice*

\[\{\tau_1\} \times \mathbb{H}\]

*gives picture of action of $SL_2(R)$ on $\Omega_{M_2}$*

---

**Slice of $H_D$**

Points of $W_D$

Points of $P_D$

Golden table

---

**Slice of Hilbert modular surface**

$D=5$

$q = Q | \{\tau_1\} \times \mathbb{H}$

---

**Exotic leaves**

---
Möller-Zagier formula

\[ Q = \left( \prod_{m \text{ odd}} \frac{d \vartheta_m(\tau, 0)}{dz} \right) \left/ \prod_{m \text{ even}} \vartheta_m(\tau, 0) \right. d\tau_1^{-1} d\tau_2^2. \]

-products taken over spin strs \( m \)
- (6 odd, 10 even)

\( (Q) = W_D - P_D \) on the Hilbert modular surface \( X_D = \mathbb{H} \times \mathbb{H} / \text{SL}_2(\mathcal{O}_D) \)