Entropy, Algebraic Integers, and moduli of surfaces

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Algebraic integers

What is the smallest integer \( \lambda > 1 \)?

M(\( \lambda \)) = product of conjugates with \(|\lambda_i| > 1\)

Mahler measure

Lehmer's Number

Lehmer's number \( \lambda = 1.176280... \)

\[ P_{10}(x) = x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1 \]

Smallest Salem Numbers, by Degree

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>( P_d(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda_2 )</td>
<td>( x^2 - 3x + 1 )</td>
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<tr>
<td>( \lambda_4 )</td>
<td>( x^4 - x^3 - x^2 - x + 1 )</td>
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<td>( \lambda_6 )</td>
<td>( x^6 - x^4 - x^3 - x^2 + 1 )</td>
</tr>
<tr>
<td>( \lambda_8 )</td>
<td>( x^8 - x^5 - x^4 - x^3 + 1 )</td>
</tr>
<tr>
<td>( \lambda_{10} )</td>
<td>( x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1 )</td>
</tr>
<tr>
<td>( \lambda_{12} )</td>
<td>( x^{12} - x^{11} + x^{10} - x^9 - x^8 - x^7 + x^2 - x + 1 )</td>
</tr>
<tr>
<td>( \lambda_{14} )</td>
<td>( x^{14} - x^{11} - x^{10} + x^7 - x^4 - x^3 + 1 )</td>
</tr>
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</table>

Conjecture (Lehmer) \( \lambda_{10} = \inf M(\alpha) \) over all algebraic integers with \( M(\alpha) > 0 \).
Lehmer's polynomial = $\det(xI-w)$ for $E_{10}$

Theorem. The spectral radius of any $w$ in any Coxeter group satisfies $r(w) = 1$ or $r(w) \geq \lambda_{10} > 1$.

Proof: uses Hilbert metric on the Tits cone.

Dynamics

What is the simplest interesting dynamical system?

Entropy

Entropy of English = $h = \approx \log 3$ (or less)

Schneier, Applied Cryptography, 1996

Number of possible English books with $N$ characters is about $3^N$ (not $26^N$)

$X$ compact, $f : X \to X$ continuous

$h(f) = \log \lambda \Leftrightarrow$

$|\{\text{orbit patterns of length } N\}| \sim \lambda^N$.

Torus examples

$X = \text{torus } \mathbb{R}^n/\mathbb{Z}^n$

$f : X \to X$ linear map induced by $A$ in $\text{GL}_n(\mathbb{Z})$

$h(f) = \log (\text{product of eigenvalues of } A \text{ with } |\lambda| > 1)$

$= \log [\text{spectral radius of } f' | H^*(X)]$

Lehmer's conjecture $\Leftrightarrow$

$h(f) \geq \log \lambda_{10}$ if $h(f) > 0$. 
Entropy on Complex Surfaces

X = compact complex manifold, dim = 2, 
f : X → X holomorphic

What small values can h(f) assume?

(dim=1 ⇒ zero entropy)

Entropy: Kähler case

Theorem (Gromov, Yomdin) The entropy of an automorphism f : X → X of a compact Kähler manifold is given by:

\[ h(f) = \log [\text{spectral radius of } f|H^*(X)] \]

Cor. For surfaces,

\[ h(f) = \log [\text{a Salem number}] = \log \rho(f|H^2(X)) \]

Complex Surfaces

Theorem (Cantat) A surface X admits an automorphism f : X → X with positive entropy only if X is birational to:

6
22
∞

- a complex torus \( \mathbb{C}^2/\Lambda \),
- a K3 surface*, or
- the projective plane \( \mathbb{P}^2 \).

(*) or Enriques

Synthesis Problem:
Salem number ⇒ surface + map

Complex torus \( \mathbb{C}^2/\Lambda \)

Theorem. There exists a 2D complex torus automorphism with h(f) = log(\( \lambda_6 \)).

(minimum possible for 2D torus)

Complement. For a projective torus, one can achieve h(f) = log(\( \lambda_4 \)) and this is optimal. (1.722 > 1.401)

Synthesis: \( f|H^1(X,\mathbb{Z}) \Rightarrow \Lambda \subset \mathbb{C}^2 \Rightarrow X \)
Rational Surfaces

\( X = \text{blowup of } \mathbb{P}^2 \text{ at } n \text{ points} \)

\[ H^2(X,\mathbb{Z}) \cong \mathbb{Z}^{1,n} \supset K_X^\perp = [E_n \text{ lattice}] \]

\[ K_X = (-3,1,1,...,1) \]

\[ \text{Aut}(X) \subset W(E_n) \quad (\text{Nagata}) \]

Theorem. The Coxeter automorphism of \( E_n \) can be realized by an automorphism \( F_n : X_n \to X_n \) of \( \mathbb{P}^2 \) blown up at \( n \) suitable points.

\( E_n \text{ spherical } \Leftrightarrow F_n \text{ periodic } \Leftrightarrow n \leq 8 \)

(Kantor, 1890s)

Example: \( F_3(x,y) = (y,y/x) \)

\[(x,y) \to (y,y/x) \to (y/x,1/x) \to (1/x,1/y) \to (1/y,1/y) \to (x/y,x) \to (x,y)\]

Lehmer's automorphism

\( F_{10} : X_{10} \to X_{10} \)

First case where \( h(F_n) > 0 \)

Theorem. The map \( F_{10} \) has minimal positive entropy among all surface automorphisms, namely \( h(F_{10}) = \log(\lambda_{10}) \).

Rational Surfaces: Synthesis

\( X = \text{blowup of } n \text{ points on a cuspidal cubic } C \text{ in } \mathbb{P}^2 \)

\[ [E_n \text{ lattice}] = \text{Pic}^0(X_n) \to \text{Pic}^0(C) = \mathbb{C} \]

Coxeter element \( w \) \quad Eigenvalue \( \lambda \) of \( w \)

\( \lambda \text{ eigenvector of } w \Rightarrow \text{positions of } n \text{ points on } C \)
**10 points on a cuspidal cubic**

\[(x,y) \mapsto (y, y/x) + (a,b)\]

Bedford-Kim

**Blowup at 11 points**

**K3 surfaces over \(\mathbb{R}\)**

\(X \subset \mathbb{R}^3\) defined by

\[(1 + x^2)(1 + y^2)(1 + z^2) + Axyz = 2\]

\(f : X \to X\) defined by

\[f = I_x \circ I_y \circ I_z\]

The map \(f\) is area-preserving!
**Complex Orbit**

**K3 surfaces: Glue results**

**Theorem.** There exists a K3 surface automorphism with $h(f) = \log(\lambda_{10})$, and this is optimal.  

*minimum possible*  

*(Oguiso - $\lambda_{14} = 1.2002$)*

**Complement.** For projective K3 surfaces, one can achieve $h(f) = \log(\lambda_6)$.  

*(1.401 > 1.176)*

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**Synthesis of a K3 automorphism with entropy $\log \lambda_{10}$**

- Coxeter automorphism of $E_{10}(a)$  
- $(3,7)$  
- $\mathcal{H}^{2,0} + \mathcal{H}^{0,2}$  
- +transcendental cycles

- Positive automorphism of $A_2 \oplus A_2$  
- $(0,4)$

- Identity factor $E_8$  
- $(0,8)$

**NS(X)**  
Signature (0,12)  
Determinant 9  
Blows down to 3 points

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**Assembly of a projective K3 automorphism with entropy $\log \lambda_6$**

- Salem factor  
- $(1,5)$  
- $\mathcal{F}_3^6$  
- Coxeter automorphism of $A_2(2)$  
- $(0,2)$

- $\mathcal{F}_3^2$  
- Coxeter automorphism of $A_{12}(a)$  
- $(2,10)$

- $\mathcal{F}_{13}$  
- Identity factor  
- $(0,2)$

**NS(X):** signature (1,9), determinant $9477 = 3^6 \cdot 13$

**Problem:** Find this example in projective space!
The moduli space $\mathcal{M}_g$ of Riemann surfaces $X$ of genus $g$ is a complex variety, dimension $3g-3$.

**Teichmüller metric:** every holomorphic map $f : \mathbb{H}^2 \to \mathcal{M}_g$ is distance-decreasing.

**Entropy on topological surfaces**

$\text{Mod}_g = \{f : \Sigma_g \to \Sigma_g\}/\text{isotopy} = \pi_1(\mathcal{M}_g)$

$h(f) = \min \{h(g) : g \text{ isotopic to } f\}$

$\geq \log \text{ spectral radius of } f^* \text{ on } H^1(\Sigma_g)$

*Stringlike Synthesis:* $f \mapsto$ a loop of Riemann surfaces

**Conjectures on finite covers**

$\Sigma_h \xrightarrow{F} \Sigma_h$

$\downarrow \quad \downarrow$

$\Sigma_g \xrightarrow{f} \Sigma_g$

$h(f) = \sup \log \text{ spectral radius of } F^* \text{ on } H^1(\Sigma_h)$, over all finite covers.

The super period map $\mathcal{M}_g \to \prod_c A_h$ is an isometry.

(from the Teichmüller metric to the Kobayashi metric)

**Kazhdan’s Theorem**

$\mathbb{H} \downarrow$

$Y \to \text{Jac}(Y) = \mathbb{C}^h/\Lambda$

$\downarrow$

$X$

*The hyperbolic metric on $X$ is the limit of the metrics inherited from the Jacobians of finite covers of $X$.***
**Counterexamples**

**Theorem.**
The super period map
\[ M_g \to \prod_c \mathcal{A}_h \]

is not an isometry in the directions coming from quadratic differentials with odd order zeros.

**Corollary.**
The entropy of most mapping classes
\[ f : \Sigma_g \to \Sigma_g \]
cannot be detected homologically, even after passing to finite covers.

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**Surface maps with genus \( \to \infty \)**
*(after Lanneau-Thiffeault, E. Hironaka, Farb-Leininger-Margalit)*

Let \( \delta_g = \exp (\text{length of shortest geodesic on } \mathcal{M}_g) \).

**Known:**
\[ \delta_1 = (3+\sqrt{5})/2 \iff \text{matrix } \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \]

\[ = \lambda_2 = \text{smallest Salem number of degree 2} \]
\[ \delta_2 = \lambda_4 = \text{smallest Salem number of degree 4} \]
\[ \delta_g = 1 + O(1/g) \quad \text{(not Salem for } g \gg 0!) \]

**Conjecture:**
\[ \lim (\delta_g)^g = \delta_1 = (3+\sqrt{5})/2 = \lambda_2 \]

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**Other fibrations** \( M^3 \to S^1 \)

\[ M^3 = S^3-L \]

**Teichmüller polynomial**
\[ H_1(M) \]

\[ \begin{array}{ccc}
-1 & & \\
& 1 & -1 \\
& & 1 \\
\end{array} \]

\[ X^2 - 3X + 1 \]

\[ T_n(X) = X^{2n} - X^n(X+1+X^{-1}) + 1 \]
**Teichmüller polynomial:**

**special values**

\[ T_2(X) = X^4 - X^3 - X^2 - X + 1 \]

⇒ genus 2 example with \( h(f) = \log \lambda_4 \)

\[ T_6(X) = X^{12} - X^7 - X^6 - X^5 + 1 \]

= \((X^2 - X + 1)(X^{10} + X^9 - X^7 - X^6 - X^5 - X^4 - X^3 + X + 1)\)

⇒ genus 5 example with \( h(f) = \log \lambda_{10} \)

\[ T_n(X) \quad ⇒ \text{genus } g \text{ examples with } h(f) \sim \log \lambda^{1/g}, \]

in agreement with conjecture

\[
\left(\frac{1 + \sqrt{5}}{2}\right)^2 \Rightarrow \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \Rightarrow \]

⇒ other fibrations ⇒ Lehmer's number in genus 5

\underline{Lehmer's number is implicit in the golden mean.}

(two shadows of the same object)