**Abstract**

We state and sketch a proof of the Riemann-Hilbert correspondence.

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1 Introduction

Consider $X = \mathbb{C}$ as a complex manifold with its structure sheaf $\mathcal{O}$ of holomorphic functions. Set $\mathcal{O}_0$ the stalk at $0$ and $K = \text{Frac}(\mathcal{O}_0)$. Then we set $\tilde{K}$ to be the ring of (possibly multivalued) holomorphic functions defined on an open, punctured disk around 0. For a matrix $A \in M_n(K)$, consider the system of ODEs

$$\frac{d}{dx} u(x) = A(x)u(x)$$

where we take solutions $u(x) \in \tilde{K}^n$. Then the set of solutions forms an $n$-dimensional vector space over $\mathbb{C}$. Pick a basis for this space of solutions, and let $S(x)$ be the matrix with the chosen basis as columns. Since the analytic continuation of $S(x)$ along a circle around $0 \in \mathbb{C}$ gives another basis of the solution space, there is an invertible matrix $G \in \text{GL}_n(\mathbb{C})$ such that

$$\lim_{t \to 2\pi} S(e^{it}x) = S(x)G$$

Thus, we have obtained (locally) from a differential equation a representation of the fundamental group of the punctured disk. One might hope for a correspondence between these data. However, if $n = 1$ and $A = P(x)$ is a polynomial, then the space of solutions is one-dimensional, generated by an entire function. (Take e.g. $u(x) = \exp(\int_0^x P(t)dt)$.) In particular, the representation of the fundamental group we obtain will be the trivial representation. Thus, we must restrict the class of differential equations we consider in order to obtain a meaningful correspondence. Classically, this leads to the notion of a regular singular point.

The Riemann-Hilbert correspondence vastly generalizes the above example. In place of differential equations on $\mathbb{C}$, we consider certain $D_X$-modules on a smooth variety $X$, and in place of representations of the fundamental group, we consider constructible sheaves on the complex manifold $X_{\text{an}}$ associated to $X$. Analogously to above, they are related by the solution functor $\text{Sol}_X$. (For computational reasons, we will prefer to use the de Rham functor $DR_X$ instead, but they are closely related by [6.5].)

**Theorem 1.1.** The de Rham functor $DR_X$ gives an equivalence of categories:

$$DR_X : D^b_{\text{rh}}(D_X) \xrightarrow{\sim} D^c_{\text{c}}(X)$$

where $D^b_{\text{rh}}(D_X)$ is the bounded derived category of $D_X$-modules consisting of complexes whose cohomology sheaves are regular, holonomic $D_X$-modules, and $D^b_{\text{c}}(X)$ is the bounded derived category of $\mathbb{C}_{X_{\text{an}}}$-modules whose cohomology sheaves are constructible.

To reach the correct conditions on $D_X$-modules to obtain an equivalence, we proceed as follows. First, we restrict to coherent $D_X$-modules, which admit well-behaved commutative approximations. This allows us to define the further condition of holonomicity. For holonomic $D_X$-modules, we are already very close: the existence of a duality functor allows us to obtain certain image functors which we did not have for general $D_X$-modules, which play a crucial role in the proof of the correspondence. Finally, as we saw above, we must impose some notion of regularity.

This exposition was written as a Minor thesis at Harvard University under the supervision of Dennis Gaitsgory. The presentation heavily follows [HTT08], and any errors introduced are my own. Unless otherwise stated, varieties in this paper are assumed to be smooth, quasi-projective varieties defined over $\mathbb{C}$. 

2
2 Algebraic $D$-modules

We begin by defining the category of $D$-modules on a smooth variety. We then define the inverse and direct image functors for $D$-modules with respect to a morphism of smooth varieties, and discuss various properties of these functors.

2.1 $D$-modules

Let $X$ be a smooth variety of dimension $n = \dim X$, $\mathcal{O}_X$ its structure sheaf, and $\Theta_X$ its tangent sheaf. We consider $\Theta_X$ and $\mathcal{O}_X$ as subsheaves of $\mathcal{E}nd_{\mathcal{C}}(\mathcal{O}_X)$, where $\Theta_X$ acts locally by derivations and $\mathcal{O}_X$ acts locally by multiplication. We then define the subsheaf $D_X$ of $\mathcal{E}nd_{\mathcal{C}}(\mathcal{O}_X)$ to be the sheaf generated by $\Theta_X$ and $\mathcal{O}_X$, and we call this the sheaf of differential operators on $X$. It will frequently be convenient to work in (affine) local coordinates on $X$.

Notation 2.1. Let $X$ be a smooth variety. We will frequently denote the restrictions $\mathcal{O}_X|_U, \Theta_X|_U, D_X|_U$ by $\mathcal{O}_U, \Theta_U, D_U$ respectively.

Example 2.2. Let $U \subset X$ be an open, affine subset of $X$. Then we can take a local coordinate system $\{x_i \in \mathcal{O}_X(U), \partial_i \in \Theta_X(U)\}_{1 \leq i \leq n}$ such that we have the following local description of $\Theta_U$:

$$\Theta_U = \bigoplus_{i=1}^{n} \mathcal{O}_U \partial_i, \quad [\partial_i, \partial_j] = 0, \quad [\partial_i, x_j] = \delta_{ij}$$

From this we obtain the following local description of $D_X$:

$$D_U = \bigoplus_{\alpha \in \mathbb{N}^n} \mathcal{O}_U \partial^\alpha, \quad \partial^\alpha = \prod_{i=1}^{n} \partial_i^{\alpha_i}$$

We call a sheaf $M$ on $X$ a left $D_X$-module if $M(U)$ is a left $D_X(U)$-module for each open $U \subset X$ and these actions commute with the restriction morphisms. The following lemma gives an equivalent characterization of left $D_X$-modules:

Lemma 2.3. Let $M$ be an $\mathcal{O}_X$-module. The data of a left $D_X$-module structure on $M$ extending the $\mathcal{O}_X$-module structure is equivalent to a $\mathcal{C}$-linear morphism

$$\nabla : \Theta_X \to \mathcal{E}nd_{\mathcal{C}}(M), \theta \mapsto \nabla_\theta$$

satisfying the following conditions:

1. $\nabla_{f\theta}(s) = f\nabla_\theta(s)$
2. $\nabla_\theta(fs) = \theta(f)s + f\nabla_\theta(s)$
3. $\nabla_{[\theta_1, \theta_2]}(s) = [\nabla_{\theta_1}, \nabla_{\theta_2}](s)$

where $f \in \mathcal{O}_X, \theta, \theta_1, \theta_2 \in \Theta_X, s \in M$ denote local sections, and the $D_X$-action is given by $\theta s = \nabla_\theta(s)$.

Proof. Given a morphism $\nabla$, the left action defined above commutes with restriction morphisms by definition. Given a left $D_X$-module $M$, the morphism $\nabla$ is given by $\theta \mapsto (s \mapsto \theta s)$, and we immediately verify the three conditions:
1. \( \nabla_{f\theta}(s) = (f\theta)s = f(\theta s) = f\nabla_\theta(s). \)
2. \( \nabla_\theta(fs) = \theta(fs) = (\theta f)s + (f\theta)s = [\theta, f]s + (f\theta)s = \theta(f)s + f\nabla_\theta(s) \)
3. \( \nabla_{[\theta_1, \theta_2]}(s) = [\theta_1, \theta_2]s = (\theta_1\theta_2)s - (\theta_2\theta_1)s = \nabla_1\circ\nabla_2(s) - \nabla_2\circ\nabla_1(s) = [\nabla_{\theta_1}, \nabla_{\theta_2}](s). \)

\[\square\]

For a locally free left \( \mathcal{O}_X \)-module \( M \) of finite rank, such a morphism \( \nabla \) as in 2.3 is called an \textit{integrable connection}, and in this situation we will refer also to \( M \) itself as an integrable connection. In what follows, it will often be useful to work with right \( D_X \)-modules as well, and we note the analogous characterization for right \( D_X \)-modules.

**Lemma 2.4.** Let \( M \) be an \( \mathcal{O}_X \)-module. The data of a right \( D_X \)-module structure on \( M \) extending the \( \mathcal{O}_X \)-module structure is equivalent to a \( \mathbb{C} \)-linear morphism

\[ \nabla : \Theta_X \to \mathcal{E}nd_{\mathbb{C}}(M), \theta \mapsto \nabla_\theta \]

satisfying the following conditions:
1. \( \nabla_{f\theta}(s) = \nabla_\theta(fs) \)
2. \( \nabla_\theta(fs) = \theta(fs) + f\nabla_\theta(s) \)
3. \( \nabla_{[\theta_1, \theta_2]}(s) = [\nabla_{\theta_1}, \nabla_{\theta_2}](s) \)

where \( f \in \mathcal{O}_X, \theta, \theta_1, \theta_2 \in \Theta_X, s \in M \) denote local sections, and the \( D_X \)-action is given by \( s\theta = -\nabla_\theta(s). \)

The proof is entirely analogous to above.

**Notation 2.5.** Let \( X \) be a smooth variety. We denote by \( \text{Mod}(D_X) \) the category of left \( D_X \)-modules, and by \( \text{Mod}(D_X^{op}) \) the category of right \( D_X \)-modules.

**Proposition 2.6.** Let \( M, N \in \text{Mod}(D_X) \) and \( M', N' \in \text{Mod}(D_X^{op}). \) Then
1. \( M \otimes_{\mathcal{O}_X} N \in \text{Mod}(D_X); \theta(s \otimes t) = \theta s \otimes t + s \otimes \theta t. \)
2. \( M' \otimes_{\mathcal{O}_X} N \in \text{Mod}(D_X^{op}); (s' \otimes t)\theta = s'\theta \otimes t - s' \otimes \theta t. \)
3. \( \mathcal{H}om_{\mathcal{O}_X}(M, N) \in \text{Mod}(D_X); (\theta \psi)(s) = \theta(\psi(s)) - \psi(\theta(s)). \)
4. \( \mathcal{H}om_{\mathcal{O}_X}(M', N') \in \text{Mod}(D_X); (\theta \psi)(s) = -\psi(s)\theta + \psi(s\theta). \)
5. \( \mathcal{H}om_{\mathcal{O}_X}(M, N') \in \text{Mod}(D_X^{op}); (\psi \theta)(s) = \psi(s\theta) + \psi(\theta(s)). \)

**Proof.** Using lemmas 2.3 and 2.4 the above can be verified by direct computation. \( \square \)

As a corollary, we obtain the following isomorphisms.

**Corollary 2.7.** Let \( M, N \in \text{Mod}(D_X) \) and \( M' \in \text{Mod}(D_X^{op}). \) Then we have isomorphisms:

\[ (M' \otimes_{\mathcal{O}_X} N) \otimes_{D_X} M \cong M' \otimes_{D_X} (M \otimes_{\mathcal{O}_X} N) \cong (M' \otimes_{\mathcal{O}_X} M) \otimes_{D_X} N \]

\[ (s' \otimes t) \otimes s \leftrightarrow s' \otimes (s \otimes t) \leftrightarrow (s' \otimes s) \otimes t \]

To translate between left and right \( D_X \)-modules, we will use the \textit{canonical sheaf} \( \Omega_X = \wedge^1 \Omega_X^1 \), where \( \Omega_X^1 \) is the sheaf of 1-forms on \( X \) (cotangent sheaf), and its \( \mathcal{O}_X \)-dual \( \Omega_X^{-1} = \mathcal{H}om_{\mathcal{O}_X}(\Omega_X, \mathcal{O}_X). \) Note that \( \Omega_X \) naturally has the structure of a right \( D_X \)-module (locally, \( \theta \in \Theta_X \) acts by the Lie derivative: \( \omega \theta = - (\text{Lie} \theta) \omega \)).
Proposition 2.8. The following functors (which we call the side-changing functors) are quasi-inverses:

\[ \Omega_X \otimes_{O_X} - : \text{Mod}(D_X) \to \text{Mod}(D^\text{op}_X) \]

\[ \Omega_X^1 \otimes_{O_X} - = \mathcal{H}\text{om}_{O_X}(\Omega_X, -) : \text{Mod}(D^\text{op}_X) \to \text{Mod}(D_X) \]

This follows from 2.6. For the rest of this paper, $D$-modules will be implicitly assumed to be left $D$-modules unless otherwise specified.

2.2 Image functors

Throughout this subsection, let $f : X \to Y$ be a morphism of smooth varieties. The main difficulty in defining the inverse and direct images of a $D_X$-module comes from the fact that a morphism $X \to Y$ does not induce an obvious relationship between $D_X$ and $D_Y$. (Morally speaking, given a morphism of commutative rings $A \to B$, there is no reason for a derivation on $A$ to induce a derivation on $B$ or vice versa.)

Inverse images

Let $M$ be a $D_Y$-module, and consider first its inverse image

\[ f^*M = O_X \otimes_{f^{-1}O_Y} f^{-1}M \]

as an $O_Y$-module. We give $f^*M$ the structure of a $D_X$-module as follows. First, consider the morphism of $O_X$-modules:

\[ O_X \otimes_{f^{-1}O_Y} f^{-1}\Omega_Y^1 \to \Omega_X^1 \]

Applying $\mathcal{H}\text{om}_{O_X}(-, O_X)$ gives a morphism

\[ \Theta_X \to O_X \otimes_{f^{-1}O_Y} f^{-1}\Theta_Y \]

which we denote by $\theta \mapsto \tilde{\theta}$. Then the action of $D_X$ on $f^*M$ is given (locally) by $\theta(\psi \otimes s) = \theta(\psi) \otimes s + \psi\tilde{\theta}(s)$. In case $M = D_Y$, we obtain a left $D_X$-module

\[ f^*D_Y = O_X \otimes_{f^{-1}O_Y} f^{-1}D_Y \]

and the right multiplication of $D_Y$ on itself induces a right $f^{-1}D_Y$-module structure on $f^*D_Y$. This bimodule will be important for defining our image functors.

Definition 2.9. The $(D_X, f^{-1}D_Y)$-bimodule $f^*D_X = O_X \otimes_{f^{-1}O_Y} f^{-1}D_Y$ obtained above is denoted by $D_{X \to Y}$.

We thus have an isomorphism $f^*M \simeq D_{X \to Y} \otimes_{f^{-1}D_Y} f^{-1}M$, from which we obtain a right-exact functor

\[ D_{X \to Y} \otimes_{f^{-1}D_Y} f^{-1} - : \text{Mod}(D_Y) \to \text{Mod}(D_X) \]

Example 2.10. We compute $D_{X \to Y}$ in the case of a closed embedding $i : X \to Y$ of smooth varieties. For $p \in X$, we may choose local coordinates $\{y_k, \partial_{y_k}\}_{1 \leq k \leq n}$ as in 2.2 on an affine open subset $p \in U \subset Y$ such that $y_{r+1} = \cdots = y_n = 0$ gives defining equations of $X$. Set $x_k = y_k \circ i$ for $1 \leq k \leq r$, giving local coordinates $\{x_k, \partial_{x_k}\}_{1 \leq k \leq r}$ for...
an affine open subset of $X$. In this situation, the morphism $\Theta_X \to \mathcal{O}_X \otimes \Theta_Y$ is given by $\partial_{x_k} \mapsto \partial_{y_k}$. Now set $D' = \oplus m_1, \ldots, m_n \mathcal{O}_Y \partial^{m_1}_{y_1} \ldots \partial^{m_n}_{y_n} \subset D_Y$. It is a subring of $D_Y$, and we have that $D_Y \simeq D' \otimes_{\mathbb{C}} \mathbb{C}[\partial_{y_{r+1}}, \ldots, \partial_{y_n}]$ as a left $D'$-module. Hence

$$D_{X \to Y} \simeq (\mathcal{O}_X \otimes \mathcal{O}_Y) \otimes \mathbb{C}[\partial_{y_{r+1}}, \ldots, \partial_{y_n}]$$

Note that in this situation $\mathcal{O}_X \otimes \mathcal{O}_Y i^{-1}D'$ is isomorphic to $D_X$.

**Direct images**

Let $M$ be a right $D_X$-module. Applying the sheaf theoretic direct image functor $f_*$ to the right $f^{-1}D_Y$-module $M \otimes_{D_X} D_{X \to Y}$ gives a right $D_Y$-module $f_*(M \otimes_{D_X} D_{X \to Y})$. Thus we have a functor

$$f_*(- \otimes_{D_X} D_{X \to Y}) : \text{Mod}(D_X) \to \text{Mod}(D_Y)$$

(We later give a slightly refined definition of this functor in terms of derived categories, in order to deal with the fact that tensoring is only right-exact while the pushforward is only left-exact.)

To obtain a direct image functor for left $D_X$-modules, we use the side-changing functors. To a left $D_X$-module $M$, we associate the following left $D_Y$-module:

$$\Omega^{-1}_{Y} \otimes_{\mathcal{O}_Y} f_*(\Omega_X \otimes_{\mathcal{O}_X} M) \otimes_{D_X} D_{X \to Y}$$

We then have an isomorphism by 2.7

$$(\Omega_X \otimes_{\mathcal{O}_X} M) \otimes_{D_X} D_{X \to Y} \cong (\Omega_X \otimes_{\mathcal{O}_X} D_{X \to Y}) \otimes_{D_X} M$$

of right $f^{-1}D_Y$-modules. Therefore, we have

$$\Omega^{-1}_{Y} \otimes_{\mathcal{O}_Y} f_*(\Omega_X \otimes_{\mathcal{O}_X} M) \otimes_{D_X} D_{X \to Y} \cong \Omega^{-1}_{Y} \otimes_{\mathcal{O}_Y} f_*((\Omega_X \otimes_{\mathcal{O}_X} D_{X \to Y}) \otimes_{D_X} M)$$

$$\cong f_*((\Omega_X \otimes_{\mathcal{O}_X} D_{X \to Y} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\Omega_{Y}^{-1}) \otimes_{D_X} M)$$

By side changing, we have that $\Omega_X \otimes_{\mathcal{O}_X} D_{X \to Y} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\Omega_{Y}^{-1}$ is a $(f^{-1}D_Y, D_X)$-bimodule. Thus we define

**Definition 2.11.** The $(f^{-1}D_Y, D_X)$-bimodule $\Omega_X \otimes_{\mathcal{O}_X} D_{X \to Y} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\Omega_{Y}^{-1}$ obtained above is denoted by $D_{Y \leftarrow X}$.

**Example 2.12.** Keeping the notation of 2.10 we obtain by similar computations a local isomorphism $D_{Y \leftarrow X} \cong \mathbb{C}[\partial_{y_{r+1}}, \ldots, \partial_{y_n}] \otimes_{\mathbb{C}} D_X$. We do not write explicitly the left $f^{-1}D_Y$-structure here.

### 2.3 $D_X$-modules as sheaves

Before moving on to derived categories of $D$-modules, we first list some basic results on $D$-modules. Most of these facts follow from the corresponding facts about affine varieties, quasi-projective varieties, and (quasi-)coherent $\mathcal{O}_X$-modules.

**Notation 2.13.** Let $X$ be a smooth variety. We denote by $\text{Mod}_{qc}(D_X)$ the category of $D_X$-modules which are quasi-coherent as $\mathcal{O}_X$-modules. We denote by $\text{Mod}(D_X)$ the category of coherent $D_X$-modules.
Proposition 2.14. Assume that $A = D_X(U)$ for some affine open subset $U \subset X$ or $A = D_{X,x}$ for some $x \in X$. Then

1. $A$ is a left (and right) notherian ring.
2. The left and right global dimensions of $A$ are at most $2 \dim X$.

Proposition 2.15. 1. $D_X$ is a coherent sheaf of rings.
2. A $D_X$-module is coherent iff it is quasi-coherent over $O_X$ and locally finitely generated over $D_X$.

We sketch a proof of the following theorem because it makes use of the $D_X$-action in a meaningful way:

Theorem 2.16. A $D_X$-module is coherent over $O_X$ iff it is an integrable connection.

**Sketch.** We sketch the forward direction. Suppose that a $M \in \text{Mod}(D_X)$ is coherent over $O_X$. Then it suffices to show that $M$ is locally free over $O_X$. By a standard fact for coherent $O_X$-modules, it is equivalent to prove that the stalk $M_x$ for any $x \in X$ is a free $O_{X,x}$-modules. For this, let us first take local coordinates $\{x_i, \partial_i\}$ around $x$ as in 2.2 such that the $m = (x_1, \ldots, x_n)$ is the maximal ideal of $O_{X,x}$.

By Nakayama’s lemma we have $s_1, \ldots, s_m \in M_x$ such that $M_x$ is generated over $O_{X,x}$ by $\{s_1, \ldots, s_m\}$, and the images of the generators $\{s_1, \ldots, s_m\}$ under the quotient $V = M_x \to M/mM_x$ form a basis of $V$ as a $O_{X,x}/m = \mathbb{C}$-module. We claim that in fact $\{s_1, \ldots, s_m\}$ are free generators of $M_x$ over $O_{X,x}$. Suppose to the contrary there is some nontrivial relation

$$\sum_{i=1}^m f_i s_i = 0$$

over $O_{X,x}$, and let $\text{ord}(f_i) = \max\{l \mid f_i \in m^l\}$. Applying $\partial_j$ to the above gives a new relation

$$0 = \sum_{i=1}^m (\partial_j f_i) s_i + f_i (\partial_j s_i) = \sum_{i=1}^m g_i s_i$$

If each term $\partial_j f_i = 0$ for all $j$ and $i$, the the original relation immediately descends to a nontrivial relation $\sum_{i=1}^m f_i s_i = 0$, so that each $f_i$ must be 0. Otherwise, because each $\partial_j s_i$ is again a $O_{X,x}$-linear combination of $\{s_1, \ldots, s_m\}$, we may pick some $j$ such that the minimum order of the $f_i$ is larger than the minimum order of the $g_i$. Repeating this argument until the minimum order reaches 0, we obtain a nontrivial relation $\sum_{i=1}^m h_i s_i = 0$ which descends to a nontrivial relation $\sum_{i=1}^m h_i s_i = 0$.

Definition 2.17. A smooth variety $X$ is called $D$-affine if

1. $\Gamma(X, -) : \text{Mod}_{qc}(D_X) \to \text{Mod}(\Gamma(X, D_X))$ is exact.
2. $\Gamma(X, M) = 0$ for $M \in \text{Mod}_{qc}(D_X) \implies M = 0$.

Note in particular that smooth, affine varieties are $D$-affine.

Proposition 2.18. Assume that $X$ is $D$-affine. Then

1. Any $M \in \text{Mod}_{qc}(D_X)$ is generated over $D_X$ by its global sections.
2. $\Gamma(X, -) : \text{Mod}_{qc}(D_X) \to \text{Mod}(\Gamma(X, D_X))$ gives an equivalence of categories.
Proposition 2.19. Assume that $X$ is $D$-affine. The equivalence
\[ \text{Mod}_{qc}(D_X) \simeq \text{Mod}(\Gamma(X, D_X)) \]
from Proposition 2.18(ii) induces an equivalence \( \text{Mod}_c(D_X) \simeq \text{Mod}(\Gamma(X, D_X)) \)

Proposition 2.20. Any \( M \in \text{Mod}_{qc}(D_X) \) can be embedded into an injective object \( I \) of \( \text{Mod}_{qc}(D_X) \) which is flabby.

Proposition 2.21. 1. A coherent $D_X$-module is globally generated by a coherent $O_X$-submodule.
2. Let $M \in \text{Mod}_{qc}(D_X)$ and $U \subset X$ open. Then any coherent $D_U$-submodule $N$ of $M|_U$ can be extended to a coherent $D_X$-submodule $\tilde{N}$ of $M$ (s.t. $\tilde{N}|_U = N$).
3. Any $M \in \text{Mod}_{qc}(D_X)$ is a union of coherent $D_X$-submodules.

Proposition 2.22. Let $X$ be a smooth quasi-projective variety. Then
1. Any $M \in \text{Mod}_{qc}(D_X)$ is a quotient of a locally free (hence locally projective hence locally flat) $D_X$-module.
2. Any $M \in \text{Mod}_c(D_X)$ is a quotient of a locally free $D_X$-module of finite rank.

Corollary 2.23. Let $X$ be a smooth quasi-projective variety. Then
1. There is a resolution $\cdots \to P_1 \to P_0 \to M \to 0$ of $M$ by locally free $D_X$-modules.
2. There is a finite resolution $0 \to P_m \to \cdots \to P_1 \to P_0 \to M \to 0$ of $M$ by locally projective $D_X$-modules.

3 Derived categories of $D$-modules

Although it might seem preferable to remain strictly within the category of $D_X$-modules, the formalism of derived categories will provide many tools to formulate the Riemann-Hilbert correspondence in complete generality. We show that the derived image functors respect composition and preserve quasi-coherence over $O_X$. In particular, this will allow us to use the strategy of decomposing a morphism $f : X \to Y$ as $X \to X \times Y \to Y$ and studying each piece separately.

3.1 Derived $D$-module categories

Notation 3.1. Let $\sharp \in \{\emptyset, +, -, b\}$. For a sheaf $R$ of rings on a topological space, we denote the derived category of $R$-modules $D^\sharp(\text{Mod}(R))$ by $D^\sharp(R)$.

Facts 3.2. Let $R$ be a sheaf of rings on a topological space $X$. Then for any $M \in \text{Mod}(R)$,
1. there an injective object $I$ of $\text{Mod}(R)$ and a monomorphism $M \to I$, and
2. there is a flat object $F$ of $\text{Mod}(R)$ and an epimorphism $F \to M$.

In particular, any complex $M_\bullet$ of $D^+(R)$ (resp. $D^-(R)$) is quasi-isomorphic to a complex $I_\bullet$ of $D^+(R)$ (resp. $F_\bullet$ of $D^-(R)$) of injective (resp. flat) $R$-modules.
Notation 3.3. Let $\sharp \in \{\emptyset, +, -, b\}$. We denote by $D^{\sharp}_{qc}(D_X)$ (resp. $D^{\sharp}_{c}(D_X)$) the subcategory of $D^b(D_X)$ consisting of complexes whose cohomology sheaves belong to $\text{Mod}_{qc}(D_X)$ (resp. $\text{Mod}_{c}(D_X)$).

Proposition 3.4. Any object of $D^b(D_X)$ (resp. $D^b_{qc}(D_X)$) is represented by a bounded complex of flat $D_X$-modules (resp. locally projective $D_X$-modules in $\text{Mod}_{qc}(D_X)$).

Proof. This follows from 2.14 and 2.23. □

Inverse images

Let $f : X \to Y$ be a morphism of smooth varieties. We can define the left derived functor of the right exact functor $f^*$:

$$Lf^* : D^b(D_Y) \to D^b(D_X), M_\bullet \mapsto D_{X \to Y} \otimes^{L}_{f^{-1}D_Y} f^{-1}M_\bullet$$

by using a flat resolution of $M_\bullet$, where $\otimes^{L}$ denotes the left derived functor of the tensor product. We call $Lf^*$ the inverse image functor.

Proposition 3.5. $Lf^*$ restricts to a functor $Lf^* : D^b_{qc}(D_Y) \to D^b_{qc}(D_X)$.

Proof. Let $M_\bullet \in D^b_{qc}(D_Y)$. By the forgetful functor $D^b(D_X) \to D^b(O_X)$, we may consider $M_\bullet \in D^b(O_Y)$. Then computing:

$$D_{X \to Y} \otimes^{L}_{f^{-1}D_Y} f^{-1}M_\bullet = (O_X \otimes^{L}_{f^{-1}O_Y} f^{-1}D_Y) \otimes^{L}_{f^{-1}D_Y} f^{-1}M_\bullet = (O_X \otimes^{L}_{f^{-1}O_Y} f^{-1}D_Y) \otimes^{L}_{f^{-1}O_Y} f^{-1}M_\bullet = O_X \otimes^{L}_{f^{-1}O_Y} f^{-1}M_\bullet$$

Then the desired result follows from the corresponding result for the functor $O_X \otimes^{L}_{f^{-1}O_Y} f^{-1} : D^b(O_Y) \to D^b(O_X)$ which follows from the fact that any $M_\bullet \in D^b_{qc}(O_Y)$ can be represented by a complex of locally free $O_Y$-modules. □

Remark 3.6. $Lf^*$ does not necessarily restrict to a functor $Lf^* : D^b_{qc}(D_Y) \to D^b_{qc}(D_X)$. For example, if $M_\bullet = D_Y$, then $Lf^* = D_{X \to Y}$, and if $f : X \to Y$ is a closed embedding with $\dim X < \dim Y$, then $D_{X \to Y}$ is a locally free $D_X$-module of infinite rank by 2.10.

We also make use of the shifted inverse image functor

$$f^\dagger = Lf^*[\dim X - \dim Y] : D^b(D_Y) \to D^b(D_X)$$

Proposition 3.7. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be morphisms of smooth varieties. Then

$$L(g \circ f)^* \simeq Lf^* \circ Lg^*, \quad (g \circ f)^\dagger \simeq f^\dagger \circ g^\dagger$$
Proof. First, we compute:

\[ D_{X \to Y} \otimes^L_{f^{-1}D_Y} f^{-1}D_Y \to Z \]

\[
= (O_X \otimes_{f^{-1}O_Y} f^{-1}D_Y) \otimes^L_{f^{-1}D_Y} f^{-1}(O_Y \otimes_{g^{-1}O_Z} g^{-1}D_Z)
\]

\[
= (O_X \otimes_{f^{-1}O_Y} f^{-1}D_Y) \otimes^L_{f^{-1}D_Y} (f^{-1}O_Y \otimes_{(gf)^{-1}O_Z} (g \circ f)^{-1}D_Z)
\]

\[
= (O_X \otimes^L_{f^{-1}O_Y} f^{-1}D_Y) \otimes^L_{f^{-1}D_Y} (f^{-1}O_Y \otimes^L_{(gf)^{-1}O_Z} (g \circ f)^{-1}D_Z)
\]

\[
\simeq O_X \otimes^L_{(gf)^{-1}O_Z} (g \circ f)^{-1}D_Z
\]

\[
= O_X \otimes_{(gf)^{-1}O_Z} (g \circ f)^{-1}D_Z
\]

\[
= D_{X \to Z}
\]

Thus, we have

\[ L(g \circ f)^* M_\bullet = D_{X \to Z} \otimes_{(gf)^{-1}O_Y} (g \circ f)^{-1}M_\bullet \]

\[
\simeq (D_{X \to Y} \otimes^L_{f^{-1}O_Y} f^{-1}D_Y \to Z) \otimes^L_{f^{-1}g^{-1}O_Y} f^{-1}g^{-1}M_\bullet
\]

\[
= D_{X \to Y} \otimes^L_{f^{-1}O_Y} f^{-1}(D_Y \to Z) \otimes^L_{g^{-1}O_Y} g^{-1}M_\bullet
\]

\[
= Lf^*(Lg^*(M_\bullet))
\]

\[ \square \]

**Proposition 3.8.** Let \( f : X \to Y \) be a smooth morphism of smooth varieties. Then

1. For \( M \in \text{Mod}(D_Y) \), we have \( H^i(Lf^*M) = 0 \) for \( i \neq 0 \).
2. For \( M \in \text{Mod}_c(D_Y) \), we have \( Lf^*M \in \text{Mod}_c(D_X) \)

**Proof.** 1. Because \( f \) is a smooth morphism, \( O_X \) is flat over \( f^{-1}O_Y \). By the proof of Proposition 3.5, \( Lf^*M \simeq O_X \otimes^L_{f^{-1}O_Y} f^{-1}M \). Thus \( Lf^*M \) has trivial cohomology for \( i \neq 0 \).

2. By 2.15 it suffices to show that the canonical morphism

\[ D_X \to D_{X \to Y} = O_X \otimes_{f^{-1}O_Y} f^{-1}D_Y, P \mapsto P(1 \otimes 1) \]

is surjective. This question is local, so we may assume \( X \) and \( Y \) to be affine. Next, we may choose coordinates \( \{ x_i, \partial x_i \}_{1 \leq i \leq n} \) on \( X \) and \( \{ y_i, \partial y_i \}_{1 \leq i \leq m} \) as in Section 2.2. Because \( f \) is smooth, these coordinates can be chosen to satisfy the additional condition that \( \partial x_i \mapsto 1 \otimes \partial y_i \) for \( 1 \leq i \leq m \) and 0 otherwise under the canonical morphism \( \Theta_X \to f^*\Theta_Y = O_X \otimes_{f^{-1}O_Y} f^{-1}\Theta_Y \). In this situation, we have that

\[
D_X \to Y = \bigoplus_{r_1, \ldots, r_m \geq 0} O_X \partial_{y_1}^{r_1} \cdots \partial_{y_m}^{r_m}
\]

and the canonical morphism \( D_X \to D_{X \to Y} \) from above is given by

\[
\partial_{x_1}^{r_1} \cdots \partial_{x_n}^{r_n} \mapsto \delta_{r_{m+1} + \cdots + r_n} \partial_{y_1}^{r_1} \cdots \partial_{y_m}^{r_m}
\]

\[ \square \]
Tensor products

The bifunctor
\[- \otimes_{\mathcal{O}_X} - : \text{Mod}(D_X) \times \text{Mod}(D_Y) \to \text{Mod}(D_X)\]
is right exact with respect to both factors, and we can thus define its left derived functor
\[- \otimes_{\mathcal{O}_X} \mathbb{L} = : D^b(D_X) \times D^b(D_Y) \to D^b(D_X)\]
by using a flat resolution of either factor.

Next let $X,Y$ be smooth varieties, and $p_1 : X \times Y \to X, p_2 : X \times Y \to Y$ the projections. For $M \in \text{Mod}(\mathcal{O}_X)$ and $N \in \text{Mod}(\mathcal{O}_Y)$, we set
\[M \boxtimes N = \mathcal{O}_{X \times Y} \otimes_{p_1^{-1}\mathcal{O}_X \otimes p_2^{-1}\mathcal{O}_Y} (p_1^{-1}M \otimes p_2^{-1}N) \in \text{Mod}(\mathcal{O}_{X \times Y})\]
This gives a bifunctor
\[- \boxtimes - : \text{Mod}(\mathcal{O}_X) \times \text{Mod}(\mathcal{O}_Y) \to \text{Mod}(\mathcal{O}_{X \times Y})\]
which is exact with respect to both factors, so extends to a functor
\[- \boxtimes - : D^b(\mathcal{O}_X) \times D^b(\mathcal{O}_Y) \to D^b(\mathcal{O}_{X \times Y})\]
For $M \in \text{Mod}(D_X)$ and $N \in \text{Mod}(D_Y)$, the $D_{X \times Y}$-module
\[D_{X \times Y} \otimes_{p_1^{-1}D_X \otimes p_2^{-1}D_Y} (p_1^{-1}M \otimes p_2^{-1}N)\]
is isomorphic as an $\mathcal{O}_{X \times Y}$-module to $M \boxtimes N$ by
\[D_{X \times Y} \simeq \mathcal{O}_{X \times Y} \otimes_{p_1^{-1}\mathcal{O}_X \otimes p_2^{-1}\mathcal{O}_Y} p_1^{-1}D_X \otimes p_2^{-1}D_Y\]
and we denote this $D_{X \times Y}$-module again by $M \boxtimes Y$, called the exterior product. We now obtain a bifunctor
\[- \boxtimes - : \text{Mod}(D_X) \times \text{Mod}(D_Y) \to \text{Mod}(D_{X \times Y})\]
which is again exact with respect to both factors, so extends to a functor
\[- \boxtimes - : D^b(D_X) \times D^b(D_Y) \to D^b(D_{X \times Y})\]

We list some facts about the exterior tensor product:

**Facts 3.9.** 1. Let $X$ and $Y$ be smooth varieties. $- \boxtimes -$ restricts to give functors $D^b_{qc}(D_X) \times D^b_{qc}(D_Y) \to D^b_{qc}(D_{X \times Y})$ and $D^b_c(D_X \times D^b_c(D_Y) \to D^b_c(D_{X \times Y})$.
2. Let $M \in \text{Mod}(D_X)$. Then $p_1^*M \simeq M \boxtimes \mathcal{O}_Y$.
3. Let $N \in \text{Mod}(D_Y)$. Then $p_2^*N \simeq \mathcal{O}_X \boxtimes N$.
4. Let $\Delta_X : X \to X \times X$ be the diagonal embedding. For $M,N \in \text{Mod}(D_X)$, we have $M \otimes_{\mathcal{O}_X} N \simeq \Delta^*_X(M \boxtimes N)$. Furthermore, for $M_*, N_* \in D^b(D_X)$, we have a canonical isomorphism $M_* \otimes_{\mathcal{O}_X}^L N_* \simeq L\Delta^*_X(M_* \boxtimes N_*)$.
5. If $P_i$ is a flat $D_{X_i}$-module for $i = 1,2$ then $P_1 \boxtimes P_2$ is a flat $D_{X_1 \times X_2}$-module.
6. Let $f_i : X_i \to Y_i$ be morphisms of smooth varieties. Then for $M_i, \bullet \in D^b(Y_i)$, we have $L(f_1 \times f_2)^*M_1, \bullet \boxtimes M_2, \bullet \simeq Lf_1^*M_1, \bullet \boxtimes Lf_2^*M_2, \bullet$.
7. Let $f : X \to Y$ be a morphism of smooth varieties. Then for $M_*, N_* \in D^b(D_Y)$, we have $Lf^*(M_* \otimes_{\mathcal{O}_Y}^L N_*) \simeq Lf^*M_* \otimes_{\mathcal{O}_X}^L Lf^*N_*$.  

11
Direct images

Let \( f : X \to Y \) be a morphism of smooth varieties. We define functors

\[
D^b(D_X) \to D^b(f^{-1}D_Y), \quad M_\bullet \mapsto D_{Y\leftarrow X} \otimes_{D_X}^L M_\bullet
\]

\[
D^b(f^{-1}D_Y) \to D^b(D_Y), \quad N_\bullet \mapsto Rf_* N_\bullet
\]

using a flat resolution of \( M_\bullet \) and an injective resolution of \( N_\bullet \). We denote the composition by

\[
\int_f : D^b(D_X) \to D^b(D_Y), \quad M_\bullet \mapsto Rf_*(D_{Y\leftarrow X} \otimes_{D_X}^L M_\bullet)
\]

and for an integer \( k \), we set \( \int_f^k M_\bullet = H^k(\int_f M_\bullet) \).

First, we recall a fact about \( Rf_* \):

**Proposition 3.10.** The functor \( Rf_* : D^b(\mathcal{O}_X) \to D^b(\mathcal{O}_Y) \) restricts to give a functor \( D^b_q(\mathcal{O}_X) \to D^b_q(\mathcal{O}_Y) \). If \( f \) is proper, it also restricts to a functor \( D^b_c(\mathcal{O}_X) \to D^b_c(\mathcal{O}_Y) \).

**Proposition 3.11.** Let \( X \xrightarrow{f} Y \xrightarrow{g} Z \) be morphisms of smooth varieties. Then we have that \( \int_{gof} = \int_g \int_f \).

**Proof.** By an analogous computation to the one in the proof of [3.7] we obtain isomorphisms

\[
D_{Z\leftarrow X} \simeq f^{-1}D_{Z\leftarrow Y} \otimes_{f^{-1}D_Y} D_{Y\leftarrow X} \simeq f^{-1}D_{Z\leftarrow Y} \otimes_{f^{-1}D_Y}^L D_{Y\leftarrow X}
\]

Hence by definition, for \( M_\bullet \in D^b(D_X) \), we obtain

\[
\int_g \int_f M_\bullet = Rg_*(D_{Z\leftarrow Y} \otimes_{D_Y}^L Rf_*(D_{Y\leftarrow X} \otimes_{D_X}^L M_\bullet))
\]

We claim that the canonical morphism

\[
F_\bullet \otimes_{D_Y}^L Rf_* G_\bullet \to Rf_*(f^{-1}F_\bullet \otimes_{f^{-1}D_Y}^L G_\bullet)
\]

is an isomorphism for any \( F_\bullet \in D^b_q(D_Y^{op}) \), \( G_\bullet \in D^b(f^{-1}D_Y) \). (The question is local, so take \( Y \) affine. Then represent \( F_\bullet \) by a complex of free right \( D_Y \)-modules, so we reduce to \( F_\bullet = D_Y^{op} \), and

\[
F_\bullet \otimes_{D_Y}^L Rf_* G_\bullet \simeq Rf_*(G_\bullet)^{\otimes I} \simeq Rf_*(G_\bullet^{\otimes I}) \simeq Rf_*(f^{-1}F_\bullet \otimes_{f^{-1}D_Y}^L G_\bullet)
\]

giving the desired isomorphism.) Hence we compute

\[
\int_g \int_f M_\bullet \simeq Rg_*Rf_*(f^{-1}D_{Z\leftarrow Y} \otimes_{f^{-1}D_Y}^L (D_{Y\leftarrow X} \otimes_{D_X}^L M_\bullet))
\]

\[
\simeq R(g \circ f)_*(((f^{-1}D_{Z\leftarrow Y} \otimes_{f^{-1}D_Y}^L D_{Y\leftarrow X}) \otimes_{D_X}^L M_\bullet)
\]

\[
\simeq R(g \circ f)_*(D_{Z\leftarrow X} \otimes_{D_X}^L M_\bullet)
\]

\[
= \int_{gof} M_\bullet
\]

\( \square \)
Proposition 3.12. Let $i : X \to Y$ be a closed embedding of smooth varieties.

1. For $M \in \text{Mod}(D_X)$, we have $\int_i^k M = 0$ for $k \neq 0$. In particular, the functor $\int_i^0 : \text{Mod}(D_X) \to \text{Mod}(D_Y)$ is exact.
2. $\int_i^0$ restricts to a functor $\int_i^0 : \text{Mod}_{qc}(D_X) \to \text{Mod}_{qc}(D_Y)$.

Sketch. By the local computation in 2.12 we have that

$$\int_i M = R\iota_* (D_Y \rightarrow X \otimes_{D_X}^L M) \simeq R\iota_* (\mathbb{C}[\partial_{y_{r+1}}, \ldots, \partial_{y_n}] \otimes_{\mathbb{C}} i_* M) \simeq \mathbb{C}[\partial_{y_{r+1}}, \ldots, \partial_{y_n}] \otimes_{\mathbb{C}} i_* M$$

This proves (i). (ii) then follows from a concrete description of the $D_Y$-module structure on $\mathbb{C}[\partial_{y_{r+1}}, \ldots, \partial_{y_n}] \otimes_{\mathbb{C}} i_* M$. \qed

For the next steps, we will need the following fact.

Lemma 3.13. There exist locally free resolutions of $\mathcal{O}_X$ as a left $D_X$-module and $\Omega_X$ as a right $D_X$-module given by

$$0 \to D_X \otimes_{\mathcal{O}_X} \wedge^n \Theta_X \to \cdots \to D_X \otimes_{\mathcal{O}_X} \wedge^0 \Theta_X \to \mathcal{O}_X \to 0$$

$$0 \to \wedge^0 \Omega_X \otimes_{\mathcal{O}_X} D_X \to \cdots \to \wedge^n \Omega^1_X \otimes_{\mathcal{O}_X} D_X \to \Omega_X \to 0$$

Let $Y, Z$ be smooth varieties, and set $X = Y \times Z$. Let $f, g : X \to Y, Z$ be the projections. To compute $D_{Y \leftarrow X} \otimes_{D_X}^L M$, we use the resolution of the right $D_X$-module $D_{Y \leftarrow X} = D_Y \boxtimes \Omega_Z$ induced by the resolution of $\Omega_Z$ from 3.13. Set $\Omega^k_{X/Y} = \mathcal{O}_Y \boxtimes \Omega^k_Z$ for $0 \leq k \leq \dim Z$. Then for $M \in \text{Mod}_{qc}(D_X)$, we define the relative de Rham complex $DR_{X/Y}(M)$ by $DR_{X/Y}(M)^k = \Omega^k_{X/Y} \otimes_{\mathcal{O}_X} M$ for $-\dim Z \leq k \leq 0$ and 0 otherwise. By construction of the relative de Rham complex, we have an equivalence $DR_{X/Y}(M) \simeq D_{Y \leftarrow X} D_{Y \leftarrow X} \otimes_{D_X}^L M$.

Proposition 3.14. Let $Y, Z$ be smooth varieties, and $f : X = Y \times Z \to Y$ the projection. Then $\int_f$ restricts to a functor $\int_f : \text{Mod}_{qc}^b(D_X) \to \text{Mod}_{qc}^b(D_Y)$.

Sketch. It suffices to show for $M \in \text{Mod}_{qc}(D_X)$ that $R^i f_* (DR_{X/Y}(M)^k)$ is quasi-coherent for any $i$ and $k$. Since $M$ is quasi-coherent over $\mathcal{O}_X$, then so is $DR_{X/Y}(M)^k$, and hence by 3.10 so is $R^i f_* (DR_{X/Y}(M)^k)$. \qed

Now we can prove:

Proposition 3.15. Let $f : X \to Y$ be a morphism of smooth varieties. Then $\int_f$ restricts to a functor $\int_f : \text{Mod}_{qc}^b(D_X) \to \text{Mod}_{qc}^b(D_Y)$.

Proof. First, factor $f : X \to Y$ into a closed embedding $X \hookrightarrow X \times Y$ followed by a projection $X \times Y \to Y$. Then by 3.11 we may assume that $f$ is either a closed embedding or a projection. The former follows from 3.12 and the latter follows from 3.14. \qed

In fact, direct images corresponding to proper morphisms even preserve coherence:

Theorem 3.16. Let $f : X \to Y$ be a proper morphism of quasi-projective varieties. Then $\int_f$ restricts to a functor $D_c^b(D_X) \to D_c^b(D_Y)$.
We mention a fact about direct images and the exterior tensor product:

**Facts 3.17.** Let \( f_i : X_i \to Y_i \) be morphisms of smooth varieties. Then for \( M_{i,\bullet} \in D^b_{qc}(D_{X_i}) \), the canonical morphism

\[
\int_{f_1} M_{1,\bullet} \boxtimes \int_{f_2} M_{2,\bullet} \to \int_{f_1 \times f_2} M_{1,\bullet} \boxtimes M_{2,\bullet}
\]

is an isomorphism.

The proofs of these statements again use the technique of splitting \( f \) into the composition of a closed embedding followed by a projection.

## 4 Coherent \( D \)-modules

So far, we have encountered the functors \( f^! \) and \( f_* \) and seen that they preserve quasi-coherence over \( \mathcal{O}_X \). However, as in remark 3.6, we saw that they do not necessarily interact nicely with coherence. In this section, we will focus on coherent \( D \)-modules, and commutative approximations of them. We then use this to find a suitable condition on coherent \( D \)-modules that is preserved by these functors.

### 4.1 Good filtrations

We first define the order filtration of \( D_X \). Recall that on any affine open \( U \subset X \), we have local coordinates \( \{ x_i, \partial_i \} \) such that \( D_U = \oplus_{\alpha \in \mathbb{N}^n} \mathcal{O}_U \partial^\alpha \). We then define \( F_i D_u = \sum_{|\alpha| < i} \mathcal{O}_U \partial^\alpha \). Then for an arbitrary open \( V \subset X \), we define

\[
F_i D_X(V) = \{ P \in D_X(V) \mid \text{res}^V_U(P) \in F_i D_X(U) \text{ for any affine open } U \subset V \}
\]

We then have the associated graded \( \text{gr}^F D_X = \bigoplus_{i=0}^{\infty} F_i D_X / F_{i-1} D_X \). We next define filtrations of \( D_X \)-modules. Let \( M \in \text{Mod}_{qc}(D_X) \). We consider a filtration of \( M \) by quasi-coherent \( \mathcal{O}_X \)-submodules \( F_i M \) satisfying

1. \( F_i M \subset F_{i+1} M \)
2. \( F_i M = 0 \) for \( i \) sufficiently small
3. \( M = \bigcup_{i \in \mathbb{Z}} F_i M \)
4. \( (F_j D_X)(F_i M) \subset F_{i+j} M \)

and define \( \text{gr}^F M = \bigoplus_{i \in \mathbb{Z}} F_i M / F_{i-1} M \) which we consider as a graded \( \text{gr}^F D_X \)-module. We give a few facts about filtered \( D_X \)-modules which will be useful later. These follow from general facts about filtered modules over rings.

**Proposition 4.1.** Let \((M,F)\) be a filtered \( D_X \)-module. TFAE:

1. \( \text{gr}^F M \) is coherent over \( \text{gr}^F D_X \).
2. \( F_i M \) is coherent over \( \mathcal{O}_X \) for each \( i \) and there is \( i_0 \) sufficiently large such that \( (F_j D_X)(F_i M) = F_{i+j} M \) for \( j \geq 0, i \geq i_0 \).
3. There is locally a surjective $D_X$-linear morphism $\Phi : D_X^{\oplus m} \to M$ and integers $n_j, 1 \leq j \leq m$ such that
$$\Phi(\bigoplus_{j=1}^m F_{i-n_j}D_X) = F_iM$$

If any of the above equivalent conditions holds for a filtered $D_X$-module $(M,F)$, then we say that $F$ is a good filtration of $M$.

**Theorem 4.2.** Any coherent $D_X$-module admits a globally defined good filtration. Conversely, a $D_X$-module with a good filtration is coherent.

Using good filtrations, we can work with commutative approximations to coherent $D$-modules. In particular, we will be able to use results directly from classical commutative algebra and algebraic geometry to study $D$-modules.

### 4.2 Characteristic varieties

As defined above, $\text{gr}^F D_X$ is a sheaf of commutative algebras finitely generated over $\mathcal{O}_X$. On an open affine $U \subset X$ with local coordinates $\{x_i, \partial_i\}$ as in 2.2, we set $\xi_i = \partial_i \text{ mod } F_0D_U \in \text{gr}_1^F D_U$. Then $\text{gr}_1^F D_U = \bigoplus_{|\alpha|=0} \mathcal{O}_U \xi^\alpha$, and $\text{gr}^F D_U = \mathcal{O}_U [\xi_1, \ldots, \xi_n]$. For $\pi : T^*X \to X$ the cotangent bundle of $X$, we may regard $\xi_1, \ldots, \xi_n$ as coordinates on the fibers of the projection over $U$, and hence we obtain an identification of $\mathcal{O}_U [\xi_1, \ldots, \xi_n]$ with $\pi_* \mathcal{O}_{T^*X} |_U$. This then gives an identification $\text{gr} D_X \simeq \pi_* \mathcal{O}_{T^*X}$.

Now let $M$ be a coherent $D_X$-module with a good filtration $F$ by 4.2. By 4.1 and the isomorphism $\text{gr}^F D_X \simeq \pi_* \mathcal{O}_{T^*X}$ obtained above, we have that $\text{gr}^F M$ is a coherent $\pi_* \mathcal{O}_{T^*X}$-module, and we set
$$\widetilde{\text{gr}^F M} = \mathcal{O}_{T^*X} \otimes_{\pi^{-1} \pi_* \mathcal{O}_{T^*X}} \pi^{-1} \text{gr}^F M$$

Then $\text{gr}^F M$ is a coherent $\mathcal{O}_{T^*X}$-module, and we call its support the characteristic variety of $M$, written $\text{Ch}(M)$. The following theorem is a consequence of a more general fact about filtered modules over filtered rings.

**Theorem 4.3.**

1. Let $M$ be a coherent $D_X$-module. Then $\text{Ch}(M)$ does not depend on the choice of a good filtration $F$.

2. For a short exact sequence $0 \to M \to N \to L \to 0$ of coherent $D_X$-modules, we have $\text{Ch}(N) = \text{Ch}(M) \cup \text{Ch}(L)$

We also remark the following difficult theorem due originally to Sato-Kawai-Kashiwara:

**Theorem 4.4.** The characteristic variety of any coherent $D_X$-module is involutive with respect to the symplectic structure of the cotangent bundle $T^*X$.

Although a proof of this theorem is beyond the scope of this paper, we note the following important corollary

**Corollary 4.5.** Let $M \in \text{Mod}_c(D_X)$. Then any irreducible component of $\text{Ch}(M)$ has dimension at least $\text{dim } X$. In particular, if $M \neq 0$, then $\text{dim } \text{Ch}(M) \geq \text{dim } X$.

If a coherent $D_X$-module $M$ has the minimal possible dimension of its characteristic variety ($\text{dim } \text{Ch}(M) \leq \text{dim } X$), then it is called holonomic.
Definition 4.8. Keeping the notation above, set \( T^*_X Y = \rho^{-1}_f T^*_X X \subset X \times_Y T^* Y \). We call a morphism \( f : X \to Y \) of smooth varieties non-characteristic with respect to a coherent \( D_Y \)-module \( M \) if \( \varpi^{-1}_f (\text{Ch}(M)) \cap T^*_X Y \subset X \times_Y T^*_Y Y \).

The following lemma can be checked by computation.

Proposition 4.6. For \( M \neq 0 \in \text{Mod}_c(D_X) \), TFAE:

1. \( M \) is coherent over \( O_X \).

2. \( \text{Ch}(M) = T^*_X X \simeq X \) (where \( T^*_X X \) denotes the zero section of \( \pi : T^* X \to X \)).

Proof. Suppose \( M \) is coherent over \( O_X \) (hence it is locally free with finite rank \( r > 0 \) by [2.16]). Then the filtration \( F \) defined by \( F_i M = 0 \) for \( i < 0 \) and \( F_i M = M \) for \( i \geq 0 \) is a good filtration on \( M \), and we have the local isomorphisms \( \text{gr}^F M \simeq M \simeq O_X^r \). Furthermore, \( \Theta_X \) acts trivially on \( \text{gr}^F M \), because \( \text{gr}^F M = 0 \) for \( l \geq 1 \), hence \( \text{Ch}(M) = T^*_X X \) is the zero section of \( \pi : T^* X \to X \).

Conversely, suppose \( \text{Ch}(M) = T^*_X X \). The problem is local on \( X \), so we may take \( X \) affine with local coords \( \{ x_i, \partial_i \}_{1 \leq i \leq n} \) as in [2.2]. In this case, \( T^* X = X \times \mathbb{C}^n \), and for any good filtration \( F \) of \( M \), we have

\[
\sqrt{\text{Ann}_{O_X[\xi_1, \ldots, \xi_n]}(\text{gr}^F M)} = (\xi_1, \ldots, \xi_n) = \mathcal{I}
\]

where \( (\xi_1, \ldots, \xi_n) \) is an ideal of \( O_X[\xi_1, \ldots, \xi_n] \). (Recall that \( \xi_i = \partial_i \text{Mod} F_0 D_X \in \text{gr}^F_0(M) \), and the identification \( \pi_* O_{T^* X} \simeq O_X[\xi_1, \ldots, \xi_n] \). Since \( I \) is noetherian, there is some \( m_0 > 0 \) such that \( I^{m_0} \subset \text{Ann}_{O_X[\xi_1, \ldots, \xi_n]}(\text{gr}^F M) \). Because \( I^{m_0} \) is generated by the monomials \( \xi^\alpha \) for \( |\alpha| = m_0 \), we thus have that

\[ \partial^\alpha F_j M \subset F_{j+m_0-1} M \]

On the other hand, because \( F \) is a good filtration, we have \( F_i D_X F_j M = F_{i+j} M \) for \( j \) sufficiently large by [4.1(ii)]. Hence

\[ F_{m_0+j} M = (F_{m_0} D_X)(F_j M) \subset F_{j+m_0-1} M \]

for \( j \) sufficiently large. Hence \( F_j M = F_{j+1} M = M \) for all \( j \) sufficiently large, so by [4.1(ii)] we have that \( M \) is a coherent \( O_X \)-module. \( \square \)

Because \( T^*_X X \simeq X \), we have \( \dim(T^*_X X) = \dim X \), so we obtain the following corollary:

Corollary 4.7. Let \( M \) be a coherent \( D_X \)-module which is also coherent over \( O_X \). (Equivalently, by [2.16] let \( M \) be an integrable connection.) Then \( M \) is holonomic.

4.3 Non-characteristic morphisms and inverse images

Although in general inverse images do not preserve coherency, we give a sufficient condition on the morphism \( f : X \to Y \) so that the inverse of a coherent \( D \)-module will be again coherent. For a morphism \( f : X \to Y \) of smooth varieties, we have the induced morphisms \( T^* X \leftarrow \rho^*_f X \times_Y T^* Y \xrightarrow{\varpi^*_f} T^* Y \).

Definition 4.8. Keeping the notation above, set \( T^*_X Y = \rho^{-1}_f T^*_X X \subset X \times_Y T^* Y \). We call a morphism \( f : X \to Y \) of smooth varieties non-characteristic with respect to a coherent \( D_Y \)-module \( M \) if \( \varpi^{-1}_f (\text{Ch}(M)) \cap T^*_X Y \subset X \times_Y T^*_Y Y \).

The following lemma can be checked by computation.
Lemma 4.9. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of smooth varieties. Then we have a commutative diagram
\[
\begin{array}{cccc}
T^*X & \xleftarrow{\rho_f} & X \times_Y T^*Y & \xleftarrow{\varphi} & X \times_Z T^*Z \\
\downarrow{\alpha_f} & & \downarrow{\psi} & & \\
T^*Y & \xleftarrow{\rho_g} & Y \times_Z T^*Z \\
\downarrow{\alpha_g} & & \\
T^*Z & \\
\end{array}
\]

where $\rho_f \circ \varphi = \rho_{gf}, \alpha_g \circ \psi = \alpha_{gf}$, and the upper right square is cartesian.

We now move on to the main theorem of this subsection.

Lemma 4.10. Let $f : X \to Y$ be an embedding of a hypersurface, non-characteristic with respect to $M \in \text{Mod}_c(D_Y)$. Then for $u \in M$, there is locally $P \in D_Y$ such that $Pu = 0$ and $f$ is non-characteristic with respect to $D_Y/D_Y P$.

Proof. We have that $\text{Ch}(D_Yu) \subset \text{Ch}(M)$, so $f$ is also non-characteristic with respect to $D_Yu$. Next, $\text{Ch}(D_Yu)$ is the zero set of $\text{gr}^Iu$, where $I = \{Q \in D_Y \mid Qu = 0\}$. Then because $T^*_XY$ is a line bundle on $X$ (in the case of a closed embedding, $T^*_XY$ is the conormal bundle of $X$ in $Y$, which in the case of a hypersurface embedding is a line bundle), we can find locally $P \in I$ such that $f$ is non-characteristic with respect to $D_Y/D_Y P$.

Theorem 4.11. Let $f : X \to Y$ be a morphism of smooth varieties non-characteristic with respect to $M \in \text{Mod}_c(D_Y)$.

1. $H^j(Lf^*M) = 0$ for all $j \neq 0$.
2. $H^0(Lf^*M)$ is a coherent $D_X$-module.
3. $\text{Ch}(H^0(Lf^*M)) \subset \rho_f \varpi_f^{-1} \text{Ch}(M)$.

Sketch. Factor $f : X \to Y$ into the composition $X \to Y \times X \to Y$. Thus we may reduce to the case where $f$ is a closed embedding of a projection. In the latter case, the assertions follow from the isomorphism $Lf^*M \simeq M \boxtimes O_Z$.

In the former case, we first consider the closed embedding of a hypersurface. To show (i), pick local coordinates $\{y_i, \partial_{y_i}\}$ on $Y$ as in [2.10] such that $y_1 = 0$ gives a defining equation for $X$, and $D_{X \to Y} \simeq D_Y/y_1D_Y$. Thus we may compute $Lf^*$ using the resolution $0 \to y_1D_Y \to D_Y \to D_{X \to Y} \to 0$. Then, $Lf^*M$ is (locally) represented by the complex $f^{-1}M \xrightarrow{\cdot y_1} f^{-1}M$ where the terms are in degree $-1$ and $0$. From here, (i) can be deduced from [4.10].

For (ii) and (iii), take a good filtration $F$ of $M$. Set $N = H^0(Lf^*M) = f^*M$, and define a filtration $F$ of $N$ by $F_iN = \text{Im}(f^*F_iM \to f^*M)$. It can then be shown that $\text{gr}^F N$ is a coherent $\text{gr}^F D_X$-module such that $\text{Ch}(N) \subset \rho_f \varpi_f^{-1} \text{Ch}(M)$.

Next, we consider when $f : X \to Y$ is a closed embedding. We proceed by induction on the codimension of $X$. For codim$_Y X = 1$, we refer to the previous case. For a more general embedding, we factor $f : X \to Y$ as a composite $X \to Z \to Y$ of closed embeddings of smooth varieties with codim$_Z X, \text{codim}_Z Y < \text{codim}_Y X$. Then using [4.9] with the induction hypothesis, we can deduce the desired results about $f$. □
4.4 Duality for $D$-modules

We have so far used filtrations to obtain commutative approximations to $D$-modules and to find a condition for inverse images to preserve coherence. In this section, we define the operation of duality, and study how it interacts with holonomicity and the image functors.

**Definition 4.12.** We define the duality functor $\mathbb{D} = \mathbb{D}_X : D^-(D_X) \to D^+(D_X)^{op}$ by

$$\mathbb{D}M_\bullet = R\mathcal{H}\text{om}_{D_X}(M_\bullet, D_X) \otimes_{O_X} \Omega_X^{-1}[d_X] = R\mathcal{H}\text{om}_{D_X}(M_\bullet, D_X \otimes_{O_X} \Omega_X^{-1}[d_X])$$

where $d_X = \dim X$.

Before we study duality, we first provide a computational lemma.

**Lemma 4.13.** Let $M \in \text{Mod}_c(D_X)$. Then for any affine open $U \subset X$,

$$\mathcal{E}\text{xt}^i_{D_X}(M, D_X)(U) = \mathcal{E}\text{xt}^i_{D_X(U)}(M(U), D_X(U))$$

To see why the operation $\mathbb{D}$ deserves the name of duality, we have the following lemma:

**Proposition 4.14.**

1. $\mathbb{D}$ sends $D^b_c(D_X)$ to $D^b_c(D_X)^{op}$
2. $\mathbb{D}^2 \simeq \text{Id}$ on $D^b_c(D_X)$.

**Sketch.**

1. First, we may assume that $M_\bullet = M \in \text{Mod}_c(D_X)$. Then we can deduce from 4.13 that $H^i(\mathbb{D}M) \in \text{Mod}_c(D_X)$ for any $i$. The boundedness follows from 4.13 and 2.14.

2. We construct a canonical morphism $M_\bullet \to \mathbb{D}^2 M_\bullet$ for $M_\bullet \in D^b(D_X)$. By definition,

$$\mathbb{D}^2 M_\bullet \simeq R\mathcal{H}\text{om}_{D_X^{op}}(R\mathcal{H}\text{om}_{D_X}(M_\bullet, D_X), D_X)$$

Now set $H_\bullet = R\mathcal{H}\text{om}_{D_X}(M_\bullet, D_X)$. Then we have

$$R\mathcal{H}\text{om}_{D_X \otimes D_X^{op}}(M_\bullet \otimes_{D_X} H_\bullet, D_X) \simeq R\mathcal{H}\text{om}_{D_X}(M_\bullet, R\mathcal{H}\text{om}_{D_X^{op}}(H_\bullet, D_X))$$

Applying $H^0(R\Gamma(X, -))$ to the above, we obtain

$$\text{Hom}_{D_X \otimes D_X^{op}}(M_\bullet \otimes_{D_X} H_\bullet, D_X) \simeq \text{Hom}_{D_X}(M_\bullet, R\mathcal{H}\text{om}_{D_X^{op}}(H_\bullet, D_X))$$

From this, we obtain our desired morphism $M \to \mathbb{D}^2 M$ by the above equivalence from the canonical morphism

$$M_\bullet \otimes_{D_X} R\mathcal{H}\text{om}_{D_X}(M_\bullet, D_X) \to D_X$$

To see that this is an isomorphism, we may first reduce to the case when $X$ is affine (hence $D$-affine), and then we may replace $M_\bullet$ with $D_X$ by 2.19 by taking a resolution $F_\bullet \simeq M_\bullet$ where $F_\bullet$ is a bounded complex of $D_X$-modules such that each term of $F_\bullet$ is a direct summand of a free $D_X$-module of finite rank. In this case, both sides are $D_X$ and the result follows immediately.
The following theorem gives further information about characteristic varieties:

**Theorem 4.15.** Let $X$ be a smooth variety and $M$ a coherent $D_X$-module.

1. $\text{codim}_{T^*X} \text{Ch}(\mathcal{E}xt^i_{D_X}(M, D_X) \otimes_{\mathcal{O}_X} \Omega_{X}^{-1}) \geq i$.
2. $\mathcal{E}xt^i_{D_X}(M, D_X) = 0$ for $i < \text{codim}_{T^*X} \text{Ch}(M)$.

Now we can state how $\mathbb{D}$ interacts with holonomicity:

**Corollary 4.16.** Let $M$ be a coherent $D_X$-module.

1. $H^i(\mathbb{D}M) = 0$ unless $-(d_X - \text{codim}_{T^*X} \text{Ch}(m)) \leq i \leq 0$.
2. $\text{codim}_{T^*X} \text{Ch}(H^i(\mathbb{D}M)) \geq d_X + i$.
3. $M$ is holonomic iff $H^i(\mathbb{D}M) = 0$ for $i \neq 0$.
4. If $M$ is holonomic, then $\mathbb{D}M \simeq H^0(\mathbb{D}M)$ is also holonomic.

**Proof.** (i) and (ii) follow immediately from 4.15 and the definition of $\mathbb{D}$. (iv) and the forward direction of (iii) follow from the (i) and (ii) and 4.5. (Note that for (iv) to make sense, we also need (iii) in 4.11.)

For the remaining direction of (iii), assume that $H^i(\mathbb{D}M) = 0$ for $i \neq 0$, and set $M^* = H^0(\mathbb{D}M)$. Then $\mathbb{D}M^* = \mathbb{D}^2 M \simeq M$, hence $H^0(\mathbb{D}M) \simeq M$ by 4.14. Then by (ii), $\dim \text{Ch}(H^0(\mathbb{D}M^*)) \geq d_X$, hence $\mathbb{D}M^* \simeq M$ is holonomic. \hfill $\square$

We now turn to relationship between duality and the image functors.

**Lemma 4.17.** For $M_\bullet \in D^b_c(D_X)$ and $N_\bullet \in D^b(D_X)$, we have

$R \mathcal{H}om_{D_X}(M_\bullet, N_\bullet) \simeq R \mathcal{H}om_{D_X}(M_\bullet, D_X) \otimes_{D_X} N_\bullet$

**Proof.** We have a canonical morphism

$\simeq R \mathcal{H}om_{D_X}(M_\bullet, D_X) \otimes_{D_X} N_\bullet \rightarrow R \mathcal{H}om_{D_X}(M_\bullet, N_\bullet)$

Thus we may assume that $M_\bullet = D_X$, in which case both side are equal to $N_\bullet$. \hfill $\square$

**Proposition 4.18.** For $M_\bullet \in D^b_c(D_X)$ and $N_\bullet \in D^b(D_X)$. We have

$R \mathcal{H}om_{D_X}(M_\bullet, N_\bullet) \simeq (\Omega_X \otimes_{\mathcal{O}_X} D_X M_\bullet) \otimes_{D_X} N_\bullet[-d_X]$  
$\simeq \Omega_X \otimes_{D_X} (D_X M_\bullet) \otimes_{\mathcal{O}_X} N_\bullet][-d_X]$  
$\simeq R \mathcal{H}om_{D_X}(\mathcal{O}_X, D_X M_\bullet \otimes_{\mathcal{O}_X} N_\bullet)$

in $D^b(\mathbb{C}_X)$. Furthermore, we have

$R \mathcal{H}om_{D_X}(\mathcal{O}_X, N_\bullet) \simeq \Omega_X \otimes_{D_X} N_\bullet[-d_X]$

**Proof.** We begin with the latter statement. By 4.17

$R \mathcal{H}om_{D_X}(\mathcal{O}_X, N_\bullet) \simeq R \mathcal{H}om_{D_X}(\mathcal{O}_X, D_X) \otimes_{D_X} N_\bullet$
Thus it suffices to compute $R\mathcal{H}om_{D_X}(\mathcal{O}_X, D_X)$, for which we use the resolution $\mathbb{3.13}$ of the left $D_X$-module $\mathcal{O}_X$ and the right $D_X$-module $\Omega_X$:

$$R\mathcal{H}om_{D_X}(\mathcal{O}_X, D_X) \simeq [\mathcal{H}om_{D_X}(D_X \otimes \mathcal{O}_X \wedge^\mathcal{H} \Theta_X, D_X)]$$

$$\simeq [\mathcal{H}om_{\mathcal{O}_X}(\wedge^\mathcal{H} \Theta_X, D_X)]$$

$$\simeq [\wedge^\mathcal{H} \Omega_X \otimes_{\mathcal{O}_X} D_X]$$

$$\simeq \Omega_X[-d_X]$$

Now let us prove the former statement. By $\mathbb{4.17}$ and the definition of $\mathbb{D}_X$:

$$R\mathcal{H}om_{D_X}(M_\bullet, N_\bullet) \simeq R\mathcal{H}om_{D_X}(M_\bullet, D_X) \otimes_{D_X} N_\bullet$$

$$\simeq (\Omega_X \otimes_{\mathcal{O}_X} \mathbb{D}_X M_\bullet) \otimes_{D_X} N_\bullet[-d_X]$$

The other isomorphisms follow from the derived version of $\mathbb{2.7}$ and the isomorphism proved in the first half of this proof.

Applying $R\Gamma(X, -)$ to the first isomorphism of $\mathbb{4.18}$

**Corollary 4.19.** Let $p : X \to \{pt\}$ be the projection. Then for $M_\bullet \in D^b_c(D_X)$ and $N_\bullet \in D^b(D_X)$, we have isomorphisms

$$R\text{Hom}_{D_X}(M_\bullet, N_\bullet) \simeq \int_p (\mathbb{D}_X M_\bullet \otimes_{D_X} N_\bullet)[-d_X]$$

$$\simeq R\text{Hom}_{D_X}(\mathcal{O}_X, \mathbb{D}_X M_\bullet \otimes_{D_X} N_\bullet)$$

**Theorem 4.20.** Let $f : X \to Y$ be a morphism of smooth varieties, and $M \in \text{Mod}_c(D_Y)$.

1. If $Lf^* M \in D^b_c(D_X)$, then there is a canonical morphism $\mathbb{D}_X Lf^* M \to Lf^* \mathbb{D}_Y M$.

2. Assume $f$ is non-characteristic with respect to $M$. Then $D_X Lf^* M \simeq Lf^* \mathbb{D}_Y M$.

We construct the morphism, but do not prove that it is an isomorphism. The proof uses the (by now) standard strategy of factoring $f$ into $X \to \mathbb{P}^n \times Y \to Y$.

**Sketch.** First, by $\mathbb{4.18}$ we have

$$\text{Hom}_{D^b(D_Y)}(M, M) \simeq \text{Hom}_{D^b(D_Y)}(\mathcal{O}_Y, \mathbb{D}_Y M \otimes_{\mathcal{O}_Y} M)$$

Applying the functor $Lf^*$ and using $\mathbb{3.9}$ we then have a morphism

$$\text{Hom}_{D^b(D_Y)}(\mathcal{O}_Y, \mathbb{D}_Y M \otimes_{\mathcal{O}_Y} M) \to \text{Hom}_{D^b(D_X)}(Lf^* \mathcal{O}_Y, Lf^* (\mathbb{D}_Y M) \otimes_{\mathcal{O}_X} Lf^* M)$$

Putting these together and then further computing:

$$\text{Hom}_{D^b(D_Y)}(M, M) \simeq \text{Hom}_{D^b(D_Y)}(\mathcal{O}_Y, \mathbb{D}_Y M \otimes_{\mathcal{O}_Y} M)$$

$$\to \text{Hom}_{D^b(D_X)}(Lf^* \mathcal{O}_Y, Lf^* (\mathbb{D}_Y M) \otimes_{\mathcal{O}_X} Lf^* M)$$

$$\simeq \text{Hom}_{D^b(D_X)}(\mathcal{O}_X, Lf^* M \otimes_{\mathcal{O}_X} Lf^* \mathbb{D}_Y M)$$

Thus we obtain a canonical morphism $\mathbb{D}_X Lf^* M \to Lf^* \mathbb{D}_Y M$ as the image under the above compositions of $\text{id}_M$. 

\[\square\]
**Theorem 4.21.** Let \( f : X \to Y \) be a proper morphism. Then we have a canonical isomorphism \( \int f \mathcal{D}_X \simeq \mathcal{D}_Y \int f : D^b_c(D_X) \to D^b_c(D_Y) \).

As above, we sketch the construction of the morphism, and omit checking that it is actually an isomorphism. The proof again uses the strategy of factoring \( f \) into \( X \to \mathbb{P}^n \times Y \to Y \).

**Sketch.** To construct the desired morphism, we will need the trace map

\[
\text{Tr}_f : \int f \mathcal{O}_X[d_X] \to \mathcal{O}_Y[d_Y]
\]

This map is constructed in two steps. First, for a closed embedding \( i : X \to Y \), we apply the canonical morphism \( \int i^! \to \text{Id} \to \mathcal{O}_Y \) gives a morphism \( \int i^! \mathcal{O}_Y \to \mathcal{O}_Y \). By local computations, \( i^! \mathcal{O}_Y = i^* \mathcal{O}_Y [d_X - d_Y] = \mathcal{O}_X [d_X - d_Y] \) and then shifting everything by \( d_Y \), we get a morphism \( \int i^! \mathcal{O}_X[d_X] \to \mathcal{O}_Y[d_Y] \). Next, for a projection \( \mathbb{P}^n \times Y \to Y \), we may reduce to the situation where \( Y \) is a single point. The desired morphism is then induced by the standard trace morphism in algebraic geometry. Finally, we obtain \( \text{Tr}_f : \int f \mathcal{O}_X[d_X] \to \mathcal{O}_Y[d_Y] \) by composing the trace morphisms for a factorization \( X \to \mathbb{P}^n \times Y \to Y \). One can then show that \( \text{Tr}_f \) does not depend on the choice of factorization and is functorial with respect to composition.

Now we construct a canonical morphism \( \int f \mathcal{D}_X \to \mathcal{D}_Y \int f \). Let \( M_* \in D^b_c(D_X) \). Computing gives

\[
\int f \mathcal{D}_X M_* = Rf_*(R \mathscr{H} \text{om}_{D_X}(M_*, D_X) \otimes^L_{D_X} D_{X \to Y}) \otimes^L_{D_Y} \Omega_{Y}^{-1}[d_Y]
= Rf_*(R \mathscr{H} \text{om}_{D_X}(M_*, D_{X \to Y})) \otimes^L_{D_Y} \Omega_{Y}^{-1}[d_X]
\]

\[
\mathcal{D}_Y \int f M_* = R \mathscr{H} \text{om}_{D_X}(\int f M_*, D_Y) \otimes^L_{D_Y} \Omega_{Y}^{-1}[d_Y]
\]

so it suffices to construct a canonical morphism

\[
\Phi(M_*) : Rf_*(R \mathscr{H} \text{om}_{D_X}(M_*, D_{X \to Y}[d_X])) \to R \mathscr{H} \text{om}_{D_Y}(\int f M_*, D_Y[d_Y])
\]

in \( D^b_c(D_Y^{op}) \). We have

\[
\int f D_{X \to Y}[d_X] = \int f Lf^* D_Y[d_X] \simeq \int f \mathcal{O}_X[d_X] \otimes^L_{D_Y} D_Y
\]

so that \( \text{Tr}_f \) induces a canonical morphism \( \int f D_{X \to Y}[d_X] \to D_Y[d_Y] \). Putting everything together, we may define \( \Phi(M_*) \) by the composition

\[
Rf_*(R \mathscr{H} \text{om}_{D_X}(M_*, D_{X \to Y}[d_X]))
\]

\[
\to Rf_* R \mathscr{H} \text{om}_{f^{-1}D_Y}(D_{Y \leftarrow X} \otimes^L_{D_X} M_*, D_{Y \leftarrow X} \otimes^L_{D_X} D_{X \to Y}[d_X])
\]

\[
\to R \mathscr{H} \text{om}_{D_Y}(Rf_* (D_{Y \leftarrow X} \otimes^L_{D_X} M_*), Rf_* (D_{Y \leftarrow X} \otimes^L_{D_X} D_{X \to Y})[d_X])
\]

\[
= R \mathscr{H} \text{om}_{D_Y}(\int f M_*, \int f D_{X \to Y}[d_X])
\]

\[
\to R \mathscr{H} \text{om}_{D_Y}(\int f M_*, D_Y[d_Y])
\]

\[\square\]
5 Holonomic $D$-modules

We now turn to holonomic $D$-modules. Although we previously required further criteria of a morphism $f : X \to Y$ for its image functors to preserve coherence, we will see that even the image functors of general morphisms $f : X \to Y$ of smooth varieties preserve holonicity. Furthermore, the duality functor will allow us to identify the previously missing left adjoints to our image functors. Finally, we also single out a particularly simple class of holonomic $D_X$-modules.

**Notation 5.1.** Let $\text{Mod}_h(D_X)$ denote the subcategory of $\text{Mod}_c(D_X)$ of holonomic $D_X$-modules, and let $D^b_c(D_X)$ denote the subcategory of $D^b_c(D_X)$ consisting of $M \in D^b_c(D_X)$ whose cohomology sheaves are holonomic.

5.1 Properties of holonomic $D$-modules

**Proposition 5.2.** 1. For an exact sequence $0 \to L \to M \to N \to 0$ in $\text{Mod}_c(D_X)$, we have $N \in \text{Mod}_h(D_X) \iff M,L \in \text{Mod}_h(D_X)$.

2. Any holonomic $D_X$-module has finite length.

**Sketch.** The first statement is an immediate corollary of [4.3]. For the second statement, we introduce an invariant called the total multiplicity, defined as follows.

Let $F$ be a good filtration of $M$, so that in particular $\text{gr}_F^i M$ is a coherent $\mathcal{O}_{T^*_X}$-module. Then for any irreducible component $C \in \text{Ch}(M)$, take an affine open $U \subset T^*_X$ such that $C \cap U = C$, and let $p_C \subset \mathcal{O}_U(U)$ be the defining ideal of $U \cap C$. Then the stalk $(\text{gr}_F^i M)_p$ is a artinian $(\mathcal{O}_{T^*_X})_p$-module (that does not depend on $U$), so has a well defined length which we denote $m_C(M)$. Then we define the total multiplicity $m(M) = \sum_{C \in \text{Ch}(M)} m_C(M)$, where the sum is over irreducible components $C$ of $\text{Ch}(M)$.

By general facts about filtered rings, $m(M) = m(L) + m(N)$ for any exact sequence $0 \to L \to M \to N \to 0$ of holonomic $D_X$-modules. Furthermore, $m(M) = 0 \iff M = 0$, so the second statement follows by induction on $m(M)$. \hfill $\square$

**Proposition 5.3.** Let $M \in \text{Mod}_h(D_X)$. Then there is an open, dense $U \subset X$ such that $M|_U$ is coherent on $\mathcal{O}_U$.

**Proof.** Let $T^*_X \subset T^*_X$ be the zero section, and set $S = \text{Ch}(M) \setminus T^*_X$. If $S = \emptyset$, then by [4.6], $M$ is coherent over $\mathcal{O}_X$. If $S \neq \emptyset$, then the fibers of the projection are at least one-dimensional, because in particular each fiber is stable under scaling by $C$, because $\text{gr}_F^i M$ is a graded module over the graded ring $\mathcal{O}_{T^*_X}$. (The grading on the latter comes from $\text{gr}_F^i D_X$.) Hence $\dim \pi(S) < \dim S \leq \dim X$. Thus there is an open subset $U \subset X$ such that $U \cap \pi(S) = \emptyset$, and thus $\text{Ch}(M|_U) \setminus T^*_U = \emptyset$, so $M|_U$ is coherent over $\mathcal{O}_U$. \hfill $\square$

We also note the following result:

**Proposition 5.4.** Let $M \in \text{Mod}_{qc}(D_X)$. For $U \subset X$ open, suppose that $N$ is a holonomic submodule of $M|_U$. Then there is a holonomic submodule $\hat{N}$ of $M$ such that $\hat{N}|_U = N$. 

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Sketch. By 2.21 we may assume $M$ coherent and $M|_U = N$. Set $L = H^0(\mathbb{D}_X M)$. Then by 4.16 we have $\text{codim}_{\mathbb{T}_X} \text{Ch}(L) \geq d_X$, hence $L$ is holonomic with $\tilde{N} = \mathbb{D}_X L$ also holonomic. By 4.14, $\tilde{N}|_U = N$, and one can check that the canonical morphism $\tilde{N} \to M$ obtained from the morphism $\mathbb{D}_X M \to L$ is injective.

**Proposition 5.5.** The duality functor $\mathbb{D}_X$ induces isomorphisms

$$\text{Mod}_h(D_X) \simeq \text{Mod}_h(D_X)^{\text{op}}, \quad D^b_h(D_X) \simeq D^b_h(D_X)^{\text{op}}$$

**Proof.** This is an immediate corollary of 4.16.

We present without proof the following fundamental result on holonomic $D$-modules.

**Theorem 5.6.** Let $f : X \to Y$ be a morphism of smooth varieties.

1. $\int f$ restricts to a functor $\int f : D^b_h(D_X) \to D^b_h(D_Y)$.
2. $f^!$ restricts to a functor $f^! : D^b_h(D_Y) \to D^b_h(D_X)$.

### 5.2 Adjunction formulas

Using duality, we can also introduce new functors:

**Definition 5.7.** Let $f : X \to Y$ be a morphism of smooth algebraic varieties. We define $\int f_!$ and $f^*$ by

$$\int f_! = \mathbb{D}_Y \int f_! \mathbb{D}_X : D^b_h(D_X) \to D^b_h(D_Y)$$

$$f^* = \mathbb{D}_X f^! \mathbb{D}_Y : D^b_h(D_Y) \to D^b_h(D_X)$$

from which we can obtain adjunction formulas:

**Theorem 5.8.** For $M_* \in D^b_h(D_X)$ and $N_* \in D^b_h(D_Y)$, we have natural isomorphisms

$$R \mathcal{H}\text{om}_{D_Y}(\int f_! M_*, N_*) \simeq Rf_* R \mathcal{H}\text{om}_{D_X}(M_*, f^! N_*)$$

$$Rf_* R \mathcal{H}\text{om}_{D_X}(f^* N_*, M_*) \simeq R \mathcal{H}\text{om}_{D_Y}(N_*, \int f_! M_*)$$
Proof. Note that either isomorphism can be deduced from the other by application of the duality functors. We prove the first:

\[ Rf_* R\mathcal{H}om_{D^b_X}(M_\bullet, f^! N_\bullet) \]
\[ \simeq Rf_* (\Omega_X \otimes_{\mathcal{O}_X} D_X X M_\bullet) \otimes_{D_X} f^! N_\bullet [-d_X] \]
\[ \simeq Rf_* (\Omega_X \otimes_{\mathcal{O}_X} D_X X M_\bullet) \otimes_{D_X} D_{X \to Y} \otimes_{f^{-1}D_Y} f^{-1} N_\bullet [d_Y] \]
\[ \simeq Rf_* (\Omega_X \otimes_{\mathcal{O}_X} D_X X M_\bullet) \otimes_{D_X} D_{X \to Y} \otimes_{D_Y} N_\bullet [-d_Y] \]
\[ \simeq (\Omega_Y \otimes_{\mathcal{O}_Y} \int_f D_X X M_\bullet) \otimes_{D_Y} N_\bullet [-d_Y] \]
\[ \simeq (\Omega_Y \otimes_{\mathcal{O}_Y} \int_f M_\bullet) \otimes_{D_Y} N_\bullet [-d_Y] \]
\[ \simeq R\mathcal{H}om_{D_Y}(\int_f M_\bullet, N_\bullet) \]

The first and last equivalences follow from 4.18 and the rest follow from definitions of the functors. \hfill \Box

From this, we obtain adjunctions:

**Corollary 5.9.** For \( M_\bullet \in D^b_X(D_X) \) and \( N_\bullet \in D^b_Y(D_Y) \), we have natural isomorphisms

\[ \Hom_{D^b_Y(D_Y)}(\int_f M_\bullet, N_\bullet) \simeq \Hom_{D^b_X(D_X)}(M_\bullet, f^! N_\bullet) \]
\[ \Hom_{D^b_Y(D_Y)}(f^* N_\bullet, M_\bullet) \simeq \Hom_{D^b_Y(D_Y)}(N_\bullet, \int_f M_\bullet) \]

*Proof.* Apply \( H^0(R\Gamma(Y, -)) \) to the isomorphisms of 5.8. \hfill \Box

**Theorem 5.10.** There is a morphism of functors \( \int_f \to \int_f : D^b_X(D_X) \to D^b_Y(D_Y) \) which is an isomorphism if \( f \) is proper.

*Sketch.* By Hironaka’s desingularization theorem, we can factor \( f : X \to Y \) as

\[ X \xrightarrow{g} X \times Y \xrightarrow{j} \tilde{X} \times Y \xrightarrow{p} Y \]

where \( \tilde{X} \) is a desingularization of \( X \), \( g, j \) are embeddings, and \( p \) is the projection. In this situation, \( g \) and \( p \) are proper, and \( j \) is an open embedding, so we may reduce to these cases.

If \( f \) is proper, then by 4.21 we have an isomorphism

\[ \int_f = \int_Y \int_f \xrightarrow{\sim} \int_f \]

If \( f \) is an open embedding, then for \( M_\bullet \in D^b_X(D_X) \), we have

\[ \Hom_{D^b_Y(D_Y)}(\int_f M_\bullet, j_! \int_f M_\bullet) \simeq \Hom_{D^b_X(D_X)}(M_\bullet, j_! \int_f M_\bullet) \]
\[ \simeq \Hom_{D^b_Y(D_Y)}(M_\bullet, M_\bullet) \]

by 5.9 and we obtain the desired morphism as the image of \( \Id \in \Hom_{D^b_X(D_X)}(M_\bullet, M_\bullet) \). \hfill \Box
5.3 Minimal extensions

A nonzero coherent $D$-module $M$ is simple if it contains no coherent $D$-submodules other than itself and 0. For any holonomic $D$-module $M$, there is a finite sequence $M = M_0 \supset \cdots \supset M_{r+1} = 0$ of holonomic $D$-submodules such that $M_i/M_{i+1}$ is simple for each $i$, by [5.2]. We now construct simple holonomic $D$-modules from locally free $D$-modules of finite rank over $\mathcal{O}$ on locally closed smooth subvarieties.

Let $Y$ be a locally closed smooth subvariety of a smooth variety $X$, and assume that the inclusion map $i : Y \hookrightarrow X$ is affine. Then $D_X \leftarrow Y$ is locally free over $D_Y$ and $Ri_* = i_*$, so for a holonomic $D_Y$-module $M$ we have $H^j \int_i M = H^j \int_{i_*} M = 0$ for $j \neq 0$, so we may thus regard $\int_i M$ and $\int_{i_*} M$ as $D_X$-modules. These are holonomic by [5.6], and by [5.10], we have a morphism $\int_{i_*} M \to \int_i M$ in $\text{Mod}_{h}(D_X)$.

We call the image $\int_{i_*} M$ of the canonical morphism $\int_{i_*} M \to \int_i M$ above the minimal extension of $M$, and $\int_{i_*} M$ is holonomic by [5.2]. We mention the following classification theorem for simple, holonomic $D$-modules.

**Theorem 5.11.**

1. Let $Y$ be a locally closed, smooth, connected subvariety of $X$ such that $i : Y \hookrightarrow X$ is affine and let $M$ be a simple holonomic $D_Y$-module. Then $L(Y, M)$ is also simple, and is the unique simple submodule of $\int_i M$.

2. Any simple holonomic $D_X$-module is isomorphic to $L(Y, M)$ for some pair $(Y, M)$, where $Y$ is as in (i) and $M$ is a simple $D_Y$-module that is locally free and of finite rank over $\mathcal{O}_Y$.

3. Let $(Y, M)$ as in (ii) and $(Y', M')$ another such pair. Then $L(Y, M) \simeq L(Y', M')$ iff $\overline{Y} = \overline{Y'}$ and $M|_U \simeq M'|_U$ for any open dense $U \subset Y \cap Y'$.

**Proposition 5.12.** Let $Y$ be a locally closed smooth subvariety of $X$ such that the embedding $i : Y \to X$ is affine, and let $M$ be an integrable connection on $Y$. Then $D_X L(Y, M) \simeq L(Y, D_Y M)$.

**Proof.** By definition of $L(Y, M)$ and exactness of the duality functor:

$$D_X L(Y, M) = D_X \text{Im}(\int_{i_*} M \to \int_i M) \simeq \text{Im}(D_X \int_{i_*} M \to D_X \int_i M)$$

and $L(Y, D_Y M) \simeq \text{Im}(\int_{i_*} D_Y M \to \int_i D_Y M)$. These are isomorphic by [4.21] and the definition of $\int_{i_*}$. \hfill $\square$

6 Analytic $D$-modules

Until this point, $X$ has denoted a smooth variety over $\mathbb{C}$. To state the Riemann-Hilbert correspondence, we will also need to study $D$-modules on complex manifolds. In this section, we give a rapid overview of the theory of $D$-modules on complex manifolds (much of which will be completely analogous to the algebraic situation).

6.1 $D$-modules on complex manifolds

Let $X$ be a complex manifold and $\mathcal{O}_X$ its sheaf of holomorphic functions. We will also need the sheaves $\Theta_X$ and $\Omega^p_X$ of holomorphic vector fields and holomorphic differential
forms of degree $p$. We define $D_X$ to be the subsheaf of $\mathcal{E}_{\text{ndC}}(\mathcal{O}_X)$ generated by $\mathcal{O}_X$ and $\Theta_X$, and we have the side-changing equivalence

$$\Omega_X \otimes \mathcal{O}_X - : \text{Mod}(D_X) \to \text{Mod}(D^p_X)$$

We also define the transfer bimodules

$$D_{X \to Y} = \mathcal{O}_X \otimes_{\mathcal{O}_Y} f^{-1} D_Y, \quad D_{Y \leftarrow X} = \Omega_X \otimes \mathcal{O}_X D_{X \to Y} \otimes_{f^{-1} \mathcal{O}_Y} f^{-1} \Omega_Y^{-1}$$

In local coordinates $\{x_i\}$ on $X$, we have $D_U = \oplus_{\alpha \in \mathbb{N}^n} \mathcal{O}_U \partial^\alpha$, where $\partial_i = \frac{\partial}{\partial x_i}$, with the order filtration defined locally by $F_i D_U = \sum_{|\alpha| \leq i} \mathcal{O}_U \partial^\alpha$. Then the associated graded $\text{gr}^F D_X$ is a sheaf of commutative algebras over $\mathcal{O}_X$, and we will often identify it with a subsheaf of $\pi_* \mathcal{O}_{T^* X}$. (We regard $\xi_i = \partial_i \in \text{gr}^F_1 D_X$ as giving coordinates on the fibers of the cotangent bundle $\pi : T^* X \to X$.)

We also have the notion of a good filtration on a $D$-module $M$. Unlike in the algebraic case, we no longer have the existence of a global good filtration, but the local version will suffice to define the characteristic variety $\text{Ch}(M)$ of a coherent $D_X$-module as follows. For an open $U \subset X$ such that $M|_U$ admits a good filtration $F$, we obtain a coherent $\mathcal{O}_{T^*U}$-module

$$\text{gr}^F(M|_U) = \mathcal{O}_{T^*U} \otimes_{\pi_U^{-1} \text{gr}^F D_U} \pi_U^{-1} \text{gr}^F M|_U$$

where $\pi_U : T^* U \to U$ is the projection. Set $\text{Ch}(M|_U)$ to be the support of the above $\mathcal{O}_{T^*U}$-module. Then $\text{Ch}(M)$ is the closed subvariety of $T^* X$ such that $\text{Ch}(M) \cap T^* U = \text{Ch}(M|_U)$ for any $U$ and $F$ as above. By an analogous argument as in the algebraic case, $\text{Ch}(M)$ is well-defined, and we have the following theorem.

**Theorem 6.1.** For any coherent $D_X$-module $M$, $\text{Ch}(M)$ is involutive with respect to the canonical symplectic structure on $T^* X$. In particular, every irreducible component of $\text{Ch}(M)$ has dimension at least $\dim X$, and $\dim \text{Ch}(M) \geq \dim X$.

We call $M$ holonomic if $\text{Ch}(M)$ has the minimal dimension $\dim X$, and we define the condition for $f$ to be non-characteristic with respect to a coherent $D_X$-module $M$ similarly to the algebraic case.

**Notation 6.2.** We denote by $\text{Mod}_{c}(D_X)$ and $\text{Mod}_{h}(D_X)$ the categories of coherent and holonomic $D_X$-modules, respectively. Furthermore, we denote by $D^b_{c}(D_X)$ and $D^b_{h}(D_X)$ the subcategories of $D^b(D_X)$ consisting of complexes whose cohomology sheaves are coherent and holonomic $D_X$-modules, respectively.

We re-introduce the various functors from before.

**Definition 6.3.** Let $f : X \to Y$ be a morphism of complex manifolds.

$$\mathbb{D}_X : D^b_c(D_X) \to D^b_c(D_X)^{\text{op}}, \quad M_* \mapsto R \mathcal{H}\text{om}_{D_X}(M_*, D_X \otimes \mathcal{O}_X \Omega_X^{-1}[d_X])$$

Note that $\mathbb{D}_X$ also preserves holonicity: $\mathbb{D}_X : D^b_h(D_X) \to D^b_h(D_X)^{\text{op}}$.

$$L f^* : D^b(D_Y) \to D^b(D_X), \quad M_* \mapsto D_{X \to Y} \otimes_{f^{-1} D_Y} f^{-1} M_*$$

$$f^! : D^b(D_Y) \to D^b(D_X), \quad M_* \mapsto L f^* M_*[d_X - d_Y]$$

$$\int_f : D^b(D_X) \to D^b(D_Y), \quad M_* \mapsto R \mathcal{H}\text{om}_{D_Y}(D_{Y \leftarrow X} \otimes D_Y M_*)$$

While many of the algebraic results carry over immediately to the analytic situation (e.g., under what conditions the functors above preserve coherence, commutivity with duality, etc.), we do not list them here.
6.2 Solution and de Rham functors

In this section, we introduce the de Rham and solution functors. These will be crucial in the Riemann-Hilbert correspondence.

**Definition 6.4.** Let $X$ be a complex manifold.
\[
DR_X : D^b(D_X) \to D^b(\mathbb{C}_X), \quad DR_X M_\bullet = \Omega_X \otimes_{\mathbb{D}_X} M_\bullet
\]
\[
Sol_X : D^b(D_X) \to D^b(\mathbb{C}_X)^{op}, \quad Sol_X M_\bullet = R\mathcal{H}om_{D_X}(M_\bullet, \mathcal{O}_X)
\]

In this section, we introduce the de Rham and solution functors. These will be crucial in the Riemann-Hilbert correspondence.

**Proposition 6.5.** For $M_\bullet \in D^b(D_X)$, we have
\[
DR_X M_\bullet \simeq R\mathcal{H}om_{D_X}(\mathcal{O}_X, M_\bullet)[d_X] \simeq Sol_X(D_X M_\bullet)[d_X]
\]

By this result, properties of $Sol_X$ can be deduced from properties of $DR_X$, and vice versa. $DR_X$ has the advantage that we can compute it using a resolution of the right $D_X$-module $\Omega_X$. Similarly to 3.13, we have a locally free resolution
\[
0 \to \Omega_X^0 \otimes_{\mathcal{O}_X} D_X \to \cdots \to \Omega_X^{d_X} \otimes_{\mathcal{O}_X} D_X \to \Omega_X \to 0
\]
so for $M \in \text{Mod}(D_X)$, we may represent $DR_X(M)[-d_X]$ in $D^b(\mathbb{C}_X)$ by the complex
\[
\Omega_X \otimes_{\mathcal{O}_X} M = [\Omega_X^0 \otimes_{\mathcal{O}_X} M \to \cdots \to \Omega_X^{d_X} \otimes_{\mathcal{O}_X} M]
\]

In the case where $M$ is an integrable connection of rank $m$ (a coherent $D_X$-module which is locally free of rank $m$ over $\mathcal{O}_X$), we have that the cohomology sheaf
\[
H^0(\Omega_X \otimes_{\mathcal{O}_X} M) \simeq \mathcal{H}om_{D_X}(\mathcal{O}_X, M)
\]
coincides with the kernel of the morphism
\[
\nabla : M \simeq \Omega_X^0 \otimes_{\mathcal{O}_X} M \to \Omega_X^1 \otimes_{\mathcal{O}_X} M
\]
whic is the sheaf
\[
M^\nabla = \{ s \in M \mid \nabla s = 0 \} = \{ 0 \in M \mid \Theta_X s = 0 \}
\]
of **horizontal sections** of the integrable connection $M$. It is a locally free $\mathbb{C}_X$-module of rank $m$. We call such $\mathbb{C}_X$-modules (locally free $\mathbb{C}_X$-modules of finite rank) **local systems**, and we denote by $\text{Loc}(X)$ the category of local systems on $X$. Conversely, given a local system $L$, we can define an integrable connection $M = \mathcal{O}_X \otimes_{\mathcal{O}_X} L$ with $\nabla : M \to \Omega_X^1 \otimes \mathbb{C}_X M$ as above given by $d \otimes \text{id}_L$. In fact, we can extend this correspondence to obtain a simple case of the Riemann-Hilbert correspondence:

**Theorem 6.6.** Let $M$ be an integrable connection of rank $m$ on a complex manifold $X$. Then $H^i(DR_X(M)) = 0$ for $i \neq -d_X$, and $H^{-d_X}(DR_X(M))$ is a local system on $X$. Thus we have an equivalence
\[
H^{-d_X}(DR_X(-)) : \text{Conn}(X) \simeq \text{Loc}(X)
\]
where $\text{Conn}(X)$ denotes the category of integrable connections on $X$.

**Theorem 6.7.** Let $f : X \to Y$ be a morphism of complex manifolds. For $M_\bullet \in D^b(D_X)$, we have an isomorphism in $D^b(\mathbb{C}_Y)$:
\[
Rf_* DR_X M_\bullet \simeq DR_Y \int_f M_\bullet
\]
If $f$ is non-characteristic with respect to a coherent $D_X$-module $M$, then we have
\[
DR_Y(Lf^* M) \simeq f^{-1} DR_X(M)[d_Y - d_X]
\]
6.3 Constructible sheaves

For a morphism \( f : X \to Y \) of analytic spaces, we have functors
\[
f^{-1} : \text{Mod}(\mathbb{C}_Y) \to \text{Mod}(\mathbb{C}_X), \quad f_*, f_! : \text{Mod}(\mathbb{C}_X) \to \text{Mod}(\mathbb{C}_Y)\]
The first is exact, and the latter two are left exact. We have their derived functors
\[
f^{-1} : D^b(\mathbb{C}_Y) \to D^b(\mathbb{C}_X), \quad Rf_*, Rf_! : D^b(\mathbb{C}_X) \to D^b(\mathbb{C}_Y)\]
and an additional functor \( f^! : D^b(\mathbb{C}_Y) \to D^b(\mathbb{C}_X) \), which is right adjoint to \( Rf_! \). Furthermore, the tensor product induces a functor
\[
- \otimes_{\mathbb{C}} - : D^b(\mathbb{C}_X) \times D^b(\mathbb{C}_X) \to D^b(\mathbb{C}_X)
\]
and we also have an exterior tensor product:

**Definition 6.8.** Let \( X \) and \( Y \) be analytic spaces. For \( K_\bullet \in D^b(\mathbb{C}_X) \) and \( L_\bullet \in D^b(\mathbb{C}_Y) \), we define
\[
K_\bullet \boxtimes \mathbb{C}L_\bullet = p_1^{-1}K_\bullet \otimes_{\mathbb{C}_X \times Y} p_2^{-1}L_\bullet
\]
where \( p_1, p_2 : X \times Y \to X, Y \) are the projections.

For an analytic space, we set \( \omega_{X, \bullet} = a_X^*\mathbb{C} \in D^b(\mathbb{C}_X) \), where \( a_X : X \to \{pt\} \) is the unique morphism to the one point space. If \( X \) is a complex manifold, then \( \omega_{X, \bullet} \simeq \mathbb{C}_X[2 \dim X] \). We define

**Definition 6.9.** Let \( X \) be a complex manifold. We define a functor
\[
D_X : D^b(\mathbb{C}_X) \to D^b(\mathbb{C}_X)^{\text{op}}, \quad D_X F_\bullet = R\mathcal{H}\text{om}_{\mathbb{C}_X}(F_\bullet, \omega_{X, \bullet})
\]
and \( D_X F_\bullet \) is called the **Verdier dual** of \( F_\bullet \in D^b(\mathbb{C}_X) \).

**Proposition 6.10.** Let \( X \) be a complex manifold. Let \( M \) be a holonomic \( D_X \)-module and \( \mathbb{D}_X M \) its dual. Then we have isomorphisms
\[
D_X(DR_X(M)) \simeq DR_X \mathbb{D}_X M, \quad D_X \text{Sol}_X(M)[d_X] \simeq \text{Sol}_X(\mathbb{D}_X M)[d_X]
\]

Let \( X \) be an analytic space. A locally finite partition \( X = \sqcup_{\alpha \in A} X_\alpha \) by locally closed analytic subsets \( X_\alpha \) is a **stratification** of \( X \) if for any \( \alpha \in A \), \( X_\alpha \) is smooth and \( \overline{X_\alpha} = \sqcup_{\beta \in B} X_\beta \) some \( B \subset A \). We call each \( X_\alpha \) a **stratum**.

**Definition 6.11.** Let \( X \) be an analytic space. A \( \mathbb{C}_X \)-module \( F \) is **constructible** on \( X \) if there is a stratification \( X = \sqcup_{\alpha \in A} X_\alpha \) such that \( F|_{X_\alpha} \) is locally free of finite rank for all \( \alpha \). If \( X \) is a variety, then a \( \mathbb{C}_{X^{an}} \)-module \( F \) is **algebraically constructible** if there is a stratification \( X = \sqcup_{\alpha \in A} X_\alpha \) such that \( F|_{X_\alpha^{an}} \) is a locally constant sheaf for all \( \alpha \).

**Notation 6.12.** For an analytic space \( X \), we denote by \( D^b_X(X) \) the subcategory of \( D^b(\mathbb{C}_X) \) consisting of complexes whose cohomology sheaves are constructible. For a variety \( X \), we denote by \( D^b_X(X) \) the subcategory of \( D^b(\mathbb{C}_{X^{an}}) \) consisting of complexes whose cohomology sheaves are algebraically constructible.

For a variety \( X \), we write by abuse of notation the sheaf \( \omega_{X^{an}, \bullet} \) and the functor \( D_{X^{an}} : D^b(\mathbb{C}_{X^{an}}) \to D^b(\mathbb{C}_{X^{an}})^{\text{op}} \) simply as \( \omega_{X, \bullet} \) and \( D_X \), respectively.

For a morphism \( f : X \to Y \) of varieties, we write \((f^{an})^{-1}, (f^{an})^!, Rf_*, Rf_! \) as \( f^{-1}, f^!, Rf_*, Rf_! \) respectively.
Theorem 6.13. 1. Let $X$ be a variety or an analytic space. Then $\omega_{\bullet} \in D^b_c(X)$, $D_X$ preserves $D^b_c(X)$, and $D^2_X \simeq \text{Id}$ on $D^b_c(X)$.

2. Let $f : X \to Y$ be a morphism of varieties or analytic spaces. Then $f^{-1}$ and $f^!$ induce $f^{-1}, f^! : D^b_c(Y) \to D^b_c(X)$, and $f^! = D_X f^{-1} D_Y$ on $D^b_c(Y)$.

3. Let $f : X \to Y$ be a morphism of varieties or analytic spaces. In the latter case, we further assume that $f$ is proper. Then $Rf_*, Rf_!$ induce $Rf_*, Rf_! : D^b_c(X) \to D^b_c(Y)$ and $Rf_! = D_Y Rf_* D_X$.

4. Let $X$ be a variety or analytic space. Then the tensor product $- \otimes \mathcal{C}$ induces $- \otimes \mathcal{C} : D^b_c(X) \times D^b_c(X) \to D^b_c(X)$.

In fact, we could have taken the above theorem as the definitions of $f!$ and $f^!$ (i.e. obtained from $f^{-1}$ and $f_*$ by Verdier duality).

Definition 6.14. Let $X$ be a variety or an analytic space. Then $F_{\bullet} \in D^b_c(X)$ is a perverse sheaf if $\dim \operatorname{supp}(H^j(F_\bullet)) \leq -j$ and $\dim \operatorname{supp}(H^j(D_X F_\bullet)) \leq -j$ for any $j \in \mathbb{Z}$. We denote by $\operatorname{Perv}(_C^X)$ the subcategory of $D^b_c(X)$ consisting of perverse sheaves.

We now mention two remarkable theorems of Kashiwara:

Theorem 6.15. Let $M$ be a holonomic $D_X$-module for $X$ a complex manifold. Then $\operatorname{Sol}(M) = R \mathcal{H} \text{om}_{D_X}(M, \mathcal{O}_X)$ and $DR_X(M) = \Omega_X \otimes_{D_X}^L M$ are objects in $D^b_c(X)$.

Theorem 6.16. Let $X$ be a complex manifold and $M$ a holonomic $D_X$-module. Then $\operatorname{Sol}(M)[d_X] = R \mathcal{H} \text{om}_{D_X}(M, \mathcal{O}_X)[d_X]$ and $DR_X(M) = \Omega_X \otimes_{D_X}^L M$ are perverse sheaves on $X$.

6.4 Analytic from algebraic

We turn now to obtaining analytic $D$-modules from algebraic $D$-modules on a smooth variety.

For an algebraic variety $X$, we denote by $X^{an}$ the corresponding analytic space. We have a morphism $i_X : (X^{an}, \mathcal{O}_{X^{an}}) \to (X, \mathcal{O}_X)$ of ringed spaces. If $X$ is a smooth variety, then $X^{an}$ is a complex manifold, and we have a morphism $i_X^{-1} D_X \to D_{X^{an}}$.

This gives a functor

$$-^{an} : \text{Mod}(D_X) \to \text{Mod}(D_{X^{an}}), \quad M \mapsto M^{an} = D_{X^{an}} \otimes_{i_X^{-1} D_X} i_X^{-1} M$$

This functor is exact because $D_{X^{an}}$ is faithfully flat over $i_X^{-1} D_X$, so the above functor is exact and extends to a functor

$$-^{an} : D^b(D_X) \to D^b(D_{X^{an}})$$

Note further that $-^{an}$ preserves coherence. We may now define the de Rham and Solution functors for a smooth variety $X$.

Definition 6.17. Let $X$ be a smooth variety. Then we define functors

$$DR_X : D^b(D_X) \to D^b(C_{X^{an}}), \quad M_{\bullet} \mapsto \Omega_{X^{an}} \otimes_{D_{X^{an}}}^L (M_{\bullet})^{an}$$

$$\operatorname{Sol}_X : D^b(D_X) \to D^b(C_{X^{an}})^{op}, \quad M_{\bullet} \mapsto R \mathcal{H} \text{om}_{D_{X^{an}}}(\langle M_{\bullet} \rangle^{an}, \mathcal{O}_{X^{an}})$$

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Proposition 6.18. Let $f : X \to Y$ be a morphism of smooth varieties. For $M_\bullet \in D^b_c(D_X)$, there is a canonical morphism $DR_Y(\int f M_\bullet) \to Rf_*(DR_X M_\bullet)$ which is an isomorphism if $f$ is proper.

Proposition 6.19. Let $X$ and $Y$ be smooth algebraic varieties. For $M_\bullet \in D^b_c(D_X)$ and $N_\bullet \in D^b_c(D_Y)$, we have a canonical morphism

$$DR_X(M_\bullet) \boxtimes_c DY_Y(N_\bullet) \to DR_{X \times Y}(M_\bullet \boxtimes N_\bullet)$$

which is an isomorphism if $M_\bullet \in D^b_c(D_X)$ or $N_\bullet \in D^b_c(D_Y)$.

Proposition 6.20. Let $X$ be a smooth variety. For $M_\bullet \in D^b_c(D_X)$, we have canonical morphisms

$$DR_X(\mathbb{D}_X M_\bullet) \to D_X(DR_X M_\bullet)$$

$$\text{Sol}_X(\mathbb{D}_X M_\bullet)[d_X] \to D_X(\text{Sol}_X M_\bullet)[d_X]$$

which are isomorphisms if $M_\bullet \in D^b_c(D_X)$.

7 Regular $D$-modules

While the classical theory of regular integrable connections on a complex manifold provides motivation for the algebraic setting, we instead take the opposite approach and begin with regular integrable connections on an algebraic variety. We then generalize to high dimensional varieties, and mention in passing an analytic result.

7.1 Regularity on curves

Let $C$ be a smooth (algebraic) curve, $p \in C$, $\mathcal{O}_{C,p}$ the local ring, and $K_{C,p}$ its fraction field.

Definition 7.1. Let $M$ be a finite dimensional $K_{C,p}$-module and $\nabla : M \to \Omega^1_{C,p} \otimes \mathcal{O}_{C,p}$ $M$ be a $\mathbb{C}$-linear map. Then $(M, \nabla)$ is called an algebraic meromorphic connection $p \in C$ if $\nabla(fu) = df \otimes u + f \nabla u$ for $f \in K_{C,p}, u \in M$.

A morphism $\varphi : (M, \nabla_M) \to (N, \nabla_N)$ of algebraic meromorphic connections at $p \in C$ is a $K_{C,p}$-linear map $\varphi : M \to N$ satisfying $\nabla_N \circ \varphi = (id \otimes \varphi) \circ \nabla_M$.

Definition 7.2. An algebraic meromorphic connection $(M, \nabla)$ at $p \in C$ is called regular if there is a finitely generated $\mathcal{O}_{C,p}$-submodule $L$ of $M$ such that $M = K_{C,p}L$ and $x\nabla(L) \subset \Omega^1_{C,p} \otimes \mathcal{O}_{C,p} L$ for some local parameter $x$ at $p$. We call such $L$ an $\mathcal{O}_{C,p}$-lattice of $(M, \nabla)$.

We now define what it means for a $D_C$-module to be reguar. Let $M$ be an integrable connection on $C$. Let $j : C \hookrightarrow \overline{C}$ be a smooth completion, and consider the $D_{\overline{C}}$-module $j_* M = \int j M$. Because $M$ was locally free over $\mathcal{O}_C$, it is free on a nonempty open $U \subset C$. We set $V = C \setminus U$, and hence $j_* M|_{\overline{C}\setminus V}$ is also free over $j_* \mathcal{O}_C|_{\overline{C}\setminus V}$. Thus $j_* M$ is locally free of finite rank over $j_* \mathcal{O}_C$. Let $p \in \overline{C} \setminus C$. Then the stalk $(j_* M)_p$ is a free module over $K_{C,p} = (j_* \mathcal{O}_C)_p$ as well as a $D_{\overline{C},p}$-module, and we have a morphism $\nabla : j_* M \to \Omega^1_{\overline{C},p} \otimes \mathcal{O}_{\overline{C},p} j_* M$ given by $m \mapsto dx \otimes \partial m$, where $x$ is a local parameter at $p$ and $\partial = \frac{d}{dx}$. We call the $D_{\overline{C}}$-module $j_* M$ the algebraic meromorphic extension of $M$.  

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Definition 7.3. Let $M$ be an integrable connection on a smooth curve $C$. For $p \in \overline{C} \setminus C$, we say that $M$ has a regular singularity at $p$ if $((j_!*M)_p, \nabla)$ as defined above is regular. $M$ is called regular if $M$ has a regular singularity at all $p \in \overline{C} \setminus C$.

Before we can define regularity for holonomic $D_C$-modules, we will need the following lemma.

Lemma 7.4. A coherent $D_C$-module $M$ is holonomic iff it is generically an integrable connection.

Proof. The forward direction follows immediately from 5.3. Conversely, suppose $M \in \text{Mod}_c(D_C)$, and there exists an open, dense $U \subset C$ such that $M|_U$ is an integrable connection. In this case, $V = C \setminus U$ is a finite set, and $\text{Ch}(M) \subset T^*_C C \cup \bigcup_{p \in V} (T^*_C)_p$, where $(T^*_C)_p$ denotes the fiber over $p$ of the projection $\pi : T^*_C C \rightarrow C$. Then because $\dim_C C = 1$ and $\dim(T^*_C)_p = 1$ and $V$ is finite, we have $\dim \text{Ch}(M) = 1$, so $M$ is holonomic.

Definition 7.5. Let $C$ be a smooth curve, and $M \in \text{Mod}_h(D_C)$. Then $M$ is regular if there is an open, dense $C_0 \subset C$ such that $M|_{C_0}$ is a regular integrable connection on $C_0$. $M_* \in D^b_h(D_C)$ is regular if all its cohomology sheaves are regular.

7.2 Regularity on general varieties

Let $X$ now denote a smooth variety, and let $j : X \hookrightarrow V$ be an open embedding of $X$ into a smooth variety $V$ such that $D = V \setminus X$ is a divisor on $V$. We set $O_V[D] = j_*O_X$; this is a coherent sheaf of rings. We call a $D_V$-module an algebraic meromorphic connection along $D$ if it is isomorphic as an $O_V$-module to a coherent $O_V[D]$-module.

Definition 7.6. An integrable connection $M$ on $X$ is regular if for any morphism $i_C : C \rightarrow X$ from a smooth curve $C$, the induced integrable connection $i_C^*M$ on $C$ is regular (as an integrable connection on a smooth curve).

Notation 7.7. We denote by $\text{Conn}(V; D)$ the category of algebraic meromorphic connections along $D$, $\text{Conn}(X)$ the category of integrable connections on $X$, and $\text{Conn}^{\text{reg}}(X)$ the subcategory of $\text{Conn}(X)$ consisting of regular integrable connections.

We now mention a few results which we will need later. Their proofs rely on results for analytic meromorphic connections, and we will not include them here.

Proposition 7.8. Let $M \in \text{Conn}(X)$. TFAE:

1. $M$ is regular.
2. There is a smooth completion $j : X \hookrightarrow \overline{X}$ such that $\overline{X} \setminus X$ is a divisor on $\overline{X}$, $(j_*M)^{\text{an}}$ is a regular analytic meromorphic connection.
3. For any smooth completion $j : X \hookrightarrow \overline{X}$ such that $\overline{X} \setminus X$ is a divisor on $\overline{X}$, $(j_*M)^{\text{an}}$ is a regular analytic meromorphic connection.

The following are due to Deligne:
Theorem 7.9. Let $D$ be a divisor on a complex manifold $X$ and $j : Y = X \setminus D \to D$ the embedding. Let $N$ be a regular (analytic) meromorphic connection along $D$. Then the following morphisms are isomorphisms
\[
DR_X(N) \to Rj_*j^{-1}DR_X(N) \\
R\Gamma(X, DR_X(N)) \to R\Gamma(Y, DR_Y(N|_Y))
\]

Theorem 7.10. Let $X$ be a smooth variety. Then the functor $\cdot^an$ induces an equivalence $\text{Conn}^{reg}(X) \to \text{Conn}(X^{an})$.

7.3 Regular holonomic $D$-modules

Finally, we define regular, holonomic $D$-modules.

Definition 7.11. Let $X$ be a smooth variety. $M \in \text{Mod}h(D_X)$ is regular if any composition factor of $M$ is isomorphic to the minimal extension $L(Y, N)$ of some regular integrable connection $N$ on a locally closed smooth subvariety $Y$ of $X$ such that the inclusion $Y \to X$ is affine.

Notation 7.12. We denote by $\text{Mod}_{rh}(D_X)$ the subcategory of $\text{Mod}_h(D_X)$ consisting of regular holonomic $D_X$-modules, and we denote by $D^b_{rh}(D_X)$ the subcategory of $D^b_{h}(D_X)$ consisting of objects whose cohomology sheaves are regular holonomic $D_X$-modules.

We now state a theorem about regular holonomic $D$-modules which will play a crucial role in the Riemann-Hilbert correspondence.

Theorem 7.13. Let $X$ be a smooth variety.

1. $D_X$ preserves $D^b_{rh}(D_X)$.

2. Let $f : X \to Y$ be a morphism of smooth varieties. Then $\int_f, \int_{f!}$ restrict to functors $D^b_{rh}(D_X) \to D^b_{rh}(D_Y)$ and $f^!, f^*$ restrict to functors $D^b_{rh}(D_Y) \to D^b_{rh}(D_X)$.

8 Riemann-Hilbert correspondence

Before we finally state the Riemann-Hilbert correspondence, we first prove a preliminary result about the interactions of the de Rham functor with the other functors we have so far.

Theorem 8.1. Let $f : X \to Y$ be a morphism of smooth varieties. Then we have the following isomorphisms of functors:
\[
D_X DR_X \simeq DR_X D_X : D^b_{h}(D_X) \to D^b_{h}(X) \\
DR_Y \circ \int_f \simeq Rf_{\ast}^{an} \circ DR_X : D^b_{rh}(D_X) \to D^b_{c}(Y) \\
DR_Y \circ \int_{f!} \simeq Rf_{!}^{an} \circ DR_X : D^b_{rh}(D_X) \to D^b_{c}(Y) \\
DR_X \circ f^! \simeq (f^{an})^! \circ DR_Y : D^b_{rh}(D_Y) \to D^b_{c}(X) \\
DR_X \circ f^* \simeq (f^{an})^{-1} \circ DR_Y : D^b_{rh}(D_Y) \to D^b_{c}(X)
\]
In the following sketch, we freely use 7.13 which is necessary even to define the functors above.

**Sketch.** The first isomorphism is [6.20] and we can immediately deduce the third and fifth isomorphisms from the first, second, and fourth using [6.13].

It remains to show the second and fourth isomorphisms. By 6.18 we have the desired morphism

\[ DR_Y \circ \int_f \to Rf_* \circ DR_X \]

and we show that it is an isomorphism when restricted to \( D_{rh}^b(D_X) \). First, we can factor \( f \) as \( X \to \overline{X} \to Y \) where the first map is an open embedding such that \( \overline{X} \setminus X \) is a normal crossings divisor on \( \overline{X} \) (by a result of Hironaka) and the second map is projective. Thus we may assume that \( f \) is an open embedding as above or projective. If \( f \) is projective, then in particular it is proper, so we have our isomorphism by 6.18.

Now let \( f \) is an open embedding as above and \( M \in \text{Mod}_{rh}(D_X) \). We proceed by induction on the length of a composition series for \( M \) (such a composition series exists by [5.2]). In this case, suffices to prove the statement for \( M \) simple, and we may reduce to the case \( M = \int_i L \) where \( i : Z \to X \) is an affine embedding of a smooth locally closed subvariety \( Z \) of \( X \) and \( L \) is a simple regular integrable connection on \( Z \). The isomorphism for \( L \) follows from 7.9 and 7.10.

Thus we may compute:

\[
DR_Y \int_f M = DR_Y \int_f \int_i L \simeq DR_Y \int_{f \circ i} \simeq R(f \circ i)_* DR_Z L \\
\simeq Rf_* Ri_* DR_Z L \simeq Rf_* DR_X \int_i L = Rf_* DR_X M
\]

For the fourth isomorphism, we first construct the desired morphism as follows.

\[
\text{Hom}_{D^b_{rh}(D_X)}(f^! N_\bullet, f^! N_\bullet) \simeq \text{Hom}_{D^b_{rh}(D_Y)}(\int_f f^! N_\bullet, N_\bullet)
\]

\[
\to \text{Hom}_{D^b_Y(Y)}(DR_Y(\int_f f^! N_\bullet), DR_Y N_\bullet)
\]

\[
\simeq \text{Hom}_{D^b_Y(Y)}(RF f_* DR_X (f^! N_\bullet), DR_Y N_\bullet)
\]

\[
\simeq \text{Hom}_{D^b_{rh}(X)}(DR_X f^! N_\bullet, f^! DR_Y N_\bullet)
\]

where the first line uses [5.9], the second line is application of \( DR_Y \), the third line comes from the isomorphism proven above, and the final line is again adjunction.

Factor \( f \) into \( X \to X \times Y \to Y \) to reduce to the cases of a closed embedding and a projection. The projection is in particular smooth, and smooth morphisms are non-characteristic for any coherent \( D_X \)-module, so the isomorphism is obtained by [6.7]. In the case of a closed embedding \( i : X \to Y \), let \( j : Y \setminus X \to Y \) be the corresponding open embedding. Then for \( N_\bullet \in D^b_{rh}(D_Y) \), we have the following morphism of distinguished triangles:

\[
\begin{align*}
DR_y \int_i i^! N_\bullet &\longrightarrow DR_Y N_\bullet \longrightarrow DR_Y \int_j j^! N_\bullet \overset{+1}{\longrightarrow} \\
Ri_* i^! DR_Y N_\bullet &\longrightarrow DR_Y N_\bullet \longrightarrow Rj_* j^! DR_Y N_\bullet \overset{+1}{\longrightarrow}
\end{align*}
\]
Since $j$ is smooth, we have that $\text{DR}_Y \int_j j^! N_* \simeq Rj_* j^! \text{DR}_Y N_*$, so that $\varphi$ is an isomorphism. Thus $\psi$ is also an isomorphism. Again by the first isomorphism, we have

$$\text{DR}_Y \int_i i^! N_* \simeq Ri_* \text{DR}_X i^! N_* ,$$

Combining this with $\psi$ gives the desired isomorphism (after precomposing with $i^{-1}$, nothing that $i^{-1} Ri_* = \text{Id}$ since $i$ is a closed embedding).  

**Theorem 8.2.** For a smooth variety $X$, the de Rham functor

$$\text{DR}_X : D^b_{rh}(D_X) \to D^b_c(X)$$

gives an equivalence of categories.

**Sketch.** First, we show that for $M_*, N_* \in D^b_{rh}(D_X)$,

$$R \text{Hom}_{D^b_{D_X}}(M_*, N_*) \simeq R \text{Hom}_{C_{\text{Xan}}}(\text{DR}_X M_*, \text{DR}_X N_*)$$

Let $\Delta : X \hookrightarrow X \times X$ be the diagonal embedding, and $p : X \to \{ pt \}$ the projection to a point. Then by 4.19, we have the equivalence

$$R \text{Hom}_{D^b_{D_X}}(M_*, N_*) \simeq \int_p \Delta^!(\mathbb{D}_X M_* \boxtimes N_*)$$

Next, we have the equivalences for $F_*, G_* \in D^b_c(X)$:

$$\Delta^!(\mathbb{D}_X F_* \boxtimes B_*) \simeq \Delta^! \mathbb{D}_{X \times X}(F_* \boxtimes \mathbb{D}_X G_*)$$

$$\simeq \mathbb{D}_X \Delta^{-1}(F \boxtimes \mathbb{D}_X G_*)$$

$$\simeq \mathbb{D}_X (F_* \boxtimes C \mathbb{D}_X G_*)$$

$$\simeq R \mathcal{H}om_{C}(F_* \boxtimes C \mathbb{D}_X G_*, \omega_{X,*})$$

$$\simeq R \mathcal{H}om_{C}(F_*, R \mathcal{H}om_{C}(\mathbb{D}_X G_*, \omega_{X,*}))$$

$$\simeq R \mathcal{H}om_{C}(F_*, \mathbb{D}_X^2 G_*)$$

$$\simeq R \mathcal{H}om_{C}(F_*, G_*)$$

and applying $Rp_* = R\Gamma(X, -)$ to the first and last terms above gives:

$$R \text{Hom}_{C_{\text{Xan}}}(F_*, G_*) \simeq Rp_* \Delta^!(\mathbb{D}_X F_* \boxtimes G_*)$$

Thus we obtain:

$$R \text{Hom}_{C_{\text{Xan}}}(\text{DR}_X M_*, \text{DR}_X N_*) \simeq Rp_* \Delta^!(\mathbb{D}_X \text{DR}_X M_* \boxtimes \text{DR}_X N_*)$$

$$6.20 \simeq Rp_* \Delta^!(\mathbb{D}_X M_* \boxtimes \text{DR}_X N_*)$$

$$6.19 \simeq Rp_* \Delta^!(\text{DR}_{X \times X}((\mathbb{D}_X M_*) \boxtimes N_*))$$

$$8.1 \simeq Rp_* \text{DR}_X (\Delta^!(\mathbb{D}_X M_* \boxtimes N_*))$$

$$8.1 \simeq \text{DR}_{pt} \int_p \Delta^!(\mathbb{D}_X M_* \boxtimes N_*)$$

$$\simeq \int_p \Delta^!(\mathbb{D}_X M_* \boxtimes N_*)$$

$$\simeq R \text{Hom}_{D^b_{D_X}}(M_*, N_*)$$
thus establishing that $DR_X$ is fully faithful. For essential surjectivity, it suffices to check on generators of $Db_c(X)$, so we may take $F_* = R\bar{i}_*L \in Db_c(C_X)$ for an affine embedding $i : \mathbb{Z} \to X$ of a locally closed smooth subvariety $Z$ of $X$ and a local system $L$ on $Z^{an}$. By [7.10] there is a regular integrable connection $N$ on $\mathbb{Z}$ such that $DR_Z N \simeq L[\dim \mathbb{Z}]$. Set $M_* = \int_i N[-\dim \mathbb{Z}] \in Db_{rh}(D_X)$. Then

$$DR_X(M_*) = DR_X \int_i N[-\dim \mathbb{Z}] \simeq R\bar{i}_*DR_Z N[-\dim \mathbb{Z}] \simeq R\bar{i}_*L = F_*$$

By [6.5] we obtain the following corollary.

**Corollary 8.3.** The solution functor

$$\text{Sol}_X : Db_{rh}(D_X) \to Db_c(X)^{op}$$

gives an equivalence of categories.

Although we will not go into detail here, we can obtain further information from the above correspondence by further investigating the category of perverse sheaves.

**Theorem 8.4.** The de Rham functor induces an equivalence

$$DR_X : \text{Mod}_{rh}(D_X) \to \text{Perv}(C_X)$$

**References**