1 Groups

1.1 The colloquial usage of the words “symmetry” and “symmetrical” is imprecise: we say, for example, that a regular pentagon

![Diagram of a regular pentagon]

is symmetrical, but what does that mean?

In the mathematical parlance, a \( \text{symmetry} \) (or more technically \( \text{automorphism} \)) of the pentagon is a (distance- and angle-preserving) transformation which leaves it unchanged. It has ten such symmetries: rotations through \( 0, \frac{2\pi}{5}, \frac{4\pi}{5}, \frac{6\pi}{5}, \) and \( \frac{8\pi}{5} \) radians, as well as reflection over any of the five lines passing through a vertex and the center of the pentagon.

These ten symmetries form a set \( D_5 \), called the \( \text{dihedral group of order 10} \). Any two elements of \( D_5 \) can be composed to obtain a third. This operation of composition has some important formal properties, summarized as follows.

**Definition 1.1.1.** A \( \text{group} \) is a set \( G \) together with an operation

\[
G \times G \to G,
\]

denoted by \( (x, y) \mapsto xy \), satisfying:

(i) there exists \( e \in G \), called the \( \text{identity} \), such that \( ex = xe = g \) for all \( x \in G \),

(ii) any \( x \in G \) has an \( \text{inverse} \) \( x^{-1} \in G \) satisfying \( xx^{-1} = x^{-1}x = e \),

(iii) for any \( x, y, z \in G \) the \( \text{associativity law} \) \( (xy)z = x(yz) \) holds.

To any mathematical (geometric, algebraic, etc.) object one can attach its \( \text{automorphism group} \) consisting of structure-preserving invertible maps from the object to itself. For example, the automorphism group of a regular pentagon is the dihedral group \( D_5 \).

For another example, let \( n \geq 0 \) and consider \( \mathbb{R}^n \) as a vector space over \( \mathbb{R} \). Linear maps \( \mathbb{R}^n \to \mathbb{R}^n \) can be thought of as \( n \times n \) matrices with coefficients in \( \mathbb{R} \), a space we denote by \( \text{Mat}_{n \times n}(\mathbb{R}) \). Composition of linear maps goes over to matrix multiplication. The linear automorphism group of \( \mathbb{R}^n \) is

\[
\text{GL}_n(\mathbb{R}) := \{ A \in \text{Mat}_{n \times n}(\mathbb{R}) \mid A \text{ is invertible} \},
\]

the \( n^{\text{th}} \) \( \text{general linear group} \) over \( \mathbb{R} \). Recall a square matrix \( A \) is invertible if and only if \( \det A \neq 0 \). We also make frequent use of the related \( n^{\text{th}} \) \( \text{special linear group} \) over \( \mathbb{R} \)

\[
\text{SL}_n(\mathbb{R}) := \{ A \in \text{Mat}_{n \times n}(\mathbb{R}) \mid \det A = 1 \}.
\]
1.2 Next we consider a geometrical figure more symmetrical than a regular pentagon, namely a circle

\[ S^1 := \{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \}. \]

We call it more symmetrical because it has infinitely many symmetries, given by the second orthogonal group

\[ O_2(\mathbb{R}) := \{ A \in \text{GL}_2(\mathbb{R}) \mid AA^\top = A^\top A = I \}. \]

Equivalently, one can define \( O_2(\mathbb{R}) \) to consist of those \( A \in O_2(\mathbb{R}) \) which preserve the dot product on \( \mathbb{R}^2 \), meaning

\[ O_2(\mathbb{R}) = \{ A \in \text{GL}_2(\mathbb{R}) \mid (Av) \cdot (Aw) = v \cdot w \text{ for all } v, w \in \mathbb{R}^2 \}. \]

This explains the term “orthogonal,” because the length of and angle between vectors can be expressed in terms of the inner product using the formulas

\[ |v| = \sqrt{v \cdot v} \text{ and } \cos \theta = \frac{v \cdot w}{|v||w|}. \]

This means that \( O_2(\mathbb{R}) \) is precisely the group of linear transformations of the plane which preserve distance and angle, the basic quantities in Euclidean geometry.

Any symmetry of a regular pentagon centered at \((0, 0)\) is also a symmetry of the circle, so we have an inclusion \( D_5 \subset O_2(\mathbb{R}) \) which realizes the former group as a subgroup of the latter. A physicist might say that the pentagon breaks the symmetry of the circle, reducing it from an infinite group to a group with ten elements.

Since the group \( O_2(\mathbb{R}) \) is infinite and it comes as a subset of \( \text{Mat}_{2 \times 2}(\mathbb{R}) \cong \mathbb{R}^4 \), we ought to think of it as a geometric figure in its own right, rather than an unstructured jumble of points. In fact, it consists of a disjoint union of two circles, which can be parameterized as

\[ \theta \mapsto \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \text{ and } \theta \mapsto \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & -\cos \theta \end{bmatrix}. \]

The first circle is the subgroup

\[ \text{SO}_2(\mathbb{R}) := \text{SL}_2(\mathbb{R}) \cap O_2(\mathbb{R}) \]

consisting of those \( A \in O_2(\mathbb{R}) \) satisfying \( \det A = 1 \), and the other component (which is not a subgroup) consists of \( A \in O_2(\mathbb{R}) \) such that \( \det A = -1 \). More geometrically, \( \text{SO}_2(\mathbb{R}) \) consists of all rotations about \((0, 0)\), while a matrix in the other component of \( O_2(\mathbb{R}) \) is a reflection over a line through \((0, 0)\).

In particular, for any \( v \in S^1 \), the map \( \text{SO}_2(\mathbb{R}) \rightarrow S^1 \) given by \( A \mapsto Av \) is bijective. We also remark that our parameterization

\[ \theta \mapsto \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \]

of \( \text{SO}_2(\mathbb{R}) \) sends addition of angles to matrix multiplication, i.e.

\[ \begin{bmatrix} \cos(\theta + \varphi) & -\sin(\theta + \varphi) \\ \sin(\theta + \varphi) & \cos(\theta + \varphi) \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{bmatrix}. \]

1.3 Although the group \( \text{GL}_2(\mathbb{R}) \) is nonabelian, its subgroup \( \text{SO}_2(\mathbb{R}) \) consisting of rotations in the plane is abelian. On the other hand, rotations in three-dimensional space form a nonabelian group. First we introduce the third orthogonal group

\[ \text{O}_3(\mathbb{R}) := \{ A \in \text{GL}_3(\mathbb{R}) \mid AA^\top = A^\top A = I \}. \]

As in the case of \( O_2(\mathbb{R}) \), a matrix \( A \in \text{O}_3(\mathbb{R}) \) satisfies \( \det A = \pm 1 \), and we put

\[ \text{SO}_3(\mathbb{R}) := \text{SL}_3(\mathbb{R}) \cap \text{O}_3(\mathbb{R}). \]

Note that reflection through a plane containing \((0, 0, 0)\) has determinant \(-1\), while a rotation about a line containing \((0, 0, 0)\) has determinant \(1\). If \( L \) is an oriented line (meaning it has a distinguished positive direction) then we write \( R^L_{\theta} \in \text{SO}_3(\mathbb{R}) \) for the rotation about \( L \) through \( \theta \) radians, counterclockwise with respect to the given orientation of \( L \) according to the right-hand rule.
Proposition 1.3.1 (Euler). Any element of $\text{SO}_3(\mathbb{R})$ has the form $R^L_\theta$ for some oriented line $L$ through $(0,0,0)$ and some angle $0 \leq \theta < 2\pi$.

Proof. Fix $A \in \text{SO}_3(\mathbb{R})$, and let $\chi_A(\lambda) = \det(\lambda I - A)$ be its characteristic polynomial. Being a polynomial in three variables with real coefficients, $\chi_A$ necessarily has a real zero (reason: its three complex roots are permuted by complex conjugation, so there can be at most one pair of conjugate non-real roots).

We claim that $A$ fixes a nonzero vector, i.e. it has 1 as an eigenvalue. If $v$ is an eigenvector of $A$ with real eigenvalue $\lambda$, then $|v| = |Av| = |\lambda v| = |\lambda||v|$ implies that $\lambda = \pm 1$. Now if $A$ has three real eigenvalues $\lambda, \mu, \nu$, then since $\det A = 1$, either $\lambda = \mu = \nu = 1$, or $-1$ occurs with multiplicity two and 1 appears once. In either case one of the eigenvalues is 1. In the remaining case, $A$ has one real eigenvalue $\lambda = \pm 1$ and two complex conjugate eigenvalues $\mu, \overline{\mu}$. Then $\lambda|\mu| = \lambda\mu\overline{\mu} = \det A = 1$ implies that $\lambda = 1$.

Let $L$ be the line spanned by an eigenvector of $A$ with eigenvalue 1, i.e. $A$ acts as the identity on $L$. Since $A$ is orthogonal, it preserves the plane $H \subset \mathbb{R}^3$ through $(0,0,0)$ perpendicular to $L$. Moreover, since $\det A = 1$ and $\det A|_L = 1$, we must have $\det L|_H = 1$. Thus $L|_H$ is a linear transformation of $H \cong \mathbb{R}^2$ which preserves the dot product and has determinant one, i.e. a rotation. 

In words, Euler’s proposition says that $\text{SO}_3(\mathbb{R})$ is precisely the subgroup of $\text{O}_3(\mathbb{R})$ consisting of rotations. It follows that any element of $\text{SO}_3(\mathbb{R})$ can be written as a composition

$$R^x_\theta, R^y_\theta, R^z_\theta,$$

where we use $R^x$ to denote rotation counterclockwise around the $x$-axis, etc. Thus, in a sense we have not rigorously defined, the group $\text{SO}_3(\mathbb{R})$ is three-dimensional. We can explicitly write

$$R^x_\theta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix}, \quad R^y_\theta = \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix}, \quad R^z_\theta = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

2 Lie algebras

2.1 The group $\text{SO}_3(\mathbb{R})$ is geometrically somewhat complicated. It is useful to have a “linear approximation” to such a group, and the appropriate object is defined as follows.

Definition 2.1.1. A Lie algebra over $\mathbb{R}$ is a vector space $\mathfrak{g}$ over $\mathbb{R}$ equipped with a bilinear operation $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ called the Lie bracket, which is required to satisfy

(i) skew-symmetry, meaning for any $x, y \in \mathfrak{g}$ we have $[x, y] = -[y, x]$;

(ii) the Jacobi identity, which says that for any $x, y, z \in \mathfrak{g}$ we have

$$[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0.$$ 

The most trivial example of a Lie algebra is a vector space $\mathfrak{g}$ with the bracket which is identically zero. Such a Lie algebra is called abelian.

For any $n \geq 1$, one can equip $\text{Mat}_{n \times n}(\mathbb{R})$ with the commutator bracket

$$[A, B] := AB - BA.$$

3
The Lie algebra $\mathfrak{gl}_n(\mathbb{R})$ so defined is called the $n^{th}$ general linear Lie algebra; it is nonabelian for $n > 1$.

We now explain the sense in which $\mathfrak{gl}_n(\mathbb{R})$ is a “linear approximation” of $\text{GL}_n(\mathbb{R})$. First, since $\text{GL}_n(\mathbb{R})$ is an open subset of the vector space $\text{Mat}_{n \times n}(\mathbb{R}) = \mathfrak{gl}_n(\mathbb{R})$, it is natural to view $\mathfrak{gl}_n(\mathbb{R})$ as the tangent space of $\text{GL}_n(\mathbb{R})$ at any point, and we may as well take that point to be the identity matrix $I \in \text{GL}_n(\mathbb{R})$. To reiterate,

$$\mathfrak{gl}_n(\mathbb{R}) = T_I(\text{GL}_n(\mathbb{R}))$$

where the notation on the right side indicates the tangent space of $\text{GL}_n(\mathbb{R})$ at $I$.

Of course $\mathfrak{gl}_n(\mathbb{R})$ contains more information than its underlying vector space, namely the commutator bracket. It is evident from the formula that this bracket measures the extent to which two matrices do not commute, and one can interpret this in terms of the group law on $\text{GL}_n(\mathbb{R})$ as follows.

We can deform $I$ in the direction of $A \in \mathfrak{gl}_n(\mathbb{R})$ by considering the matrix $I + tA$ for $t \in \mathbb{R}$. If $|t|$ is small enough then $I + tA \in \text{GL}_n(\mathbb{R})$, and in particular we have

$$(I + tA)^{-1} = I - tA + t^2 A - \cdots .$$

For $|t|$ small the quantity $t^2$ is negligible: after all, we are interested in a linear, i.e. first-order, approximation of $\text{GL}_n(\mathbb{R})$. Thus, up to first order we have

$$(I + tA)^{-1} = I - tA .$$

To quantify the extent to which $I + sA$ and $I + tB$ do not commute, we consider their commutator

$$(I + sA)(I + tB)(I + sA)^{-1}(I + tB)^{-1} = (I + sA)(I + tB)(I + sA)^{-1}(I + tB)^{-1} ,$$

which equals $I$ if and only if $I + sA$ and $I + tB$ commute.

**Exercise 2.1.2.** Show that the coefficient of $st$ in

$$(I + sA)(I + tB)(I + sA)^{-1}(I + tB)^{-1}$$

is $[A, B]$. Equivalently, we have

$$[A, B] = \left. \frac{d}{ds} \frac{d}{dt} \left( (I + sA)(I + tB)(I + sA)^{-1}(I + tB)^{-1} \right) \right|_{s,t=0} .$$

(2.1.1)

2.2 Now we explain a procedure by which we can attach a Lie algebra to a group other than $\text{GL}_n(\mathbb{R})$. First, we must understand tangent spaces more systematically.

Let $f_1(x_1, \ldots, x_n), \ldots, f_r(x_1, \ldots, x_n)$ be a collection of (infinitely) differentiable functions $\mathbb{R}^n \to \mathbb{R}$, which for us will usually be polynomials. Put

$$X := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid f_1(x_1, \ldots, x_n) = \cdots = f_r(x_1, \ldots, x_n) = 0.\}$$

and suppose that $x = (x_1, \ldots, x_n) \in X$. We define the tangent space $T_x(X) \subset \mathbb{R}^n$ to $X$ at $x$ to be the vector space of solutions to the linear equations

$$\frac{\partial f_1}{\partial x_1} \bigg|_x x_1 + \cdots + \frac{\partial f_1}{\partial x_n} \bigg|_x x_n = 0$$

$$\vdots$$

$$\frac{\partial f_r}{\partial x_1} \bigg|_x x_1 + \cdots + \frac{\partial f_r}{\partial x_n} \bigg|_x x_n = 0 .$$

Note that $x$ is not necessarily in $T_x(x)$ but 0 is, so that $x \in x + T_x(X)$. This “tangent affine space” $x + T_x(X)$ is the one we learn to draw on curves and surfaces in a calculus class.
We will only consider those $X \subset \mathbb{R}^n$ which are smooth, i.e. their tangent space has the same dimension at every $x \in X$, which we may then meaningfully call the dimension of $X$.

Suppose we are given a subgroup $G \subset \text{GL}_n(\mathbb{R})$ such that $G = \text{GL}_n(\mathbb{R}) \cap X$ for some

$$X \subset \text{Mat}_{n \times n}(\mathbb{R}) = \mathbb{R}^{n^2}$$

defined by equations as above. Then $G$ is automatically smooth, because for any $x, y \in G$, translation by $yx^{-1}$ defines an isomorphism $T_x(G) \cong T_y(G)$. The Lie algebra $\mathfrak{g}$ attached to the group $G$ has underlying vector space $T_1(G) \subset \mathfrak{gl}_n(\mathbb{R})$, with the Lie bracket inherited from $\mathfrak{gl}_n(\mathbb{R})$. To make sense of this, we must verify that $\mathfrak{g}$ is closed under the commutator bracket.

**Proposition 2.2.1.** For $A, B \in \mathfrak{g}$ where $\mathfrak{g} \subset \mathfrak{gl}_n(\mathbb{R})$ is as above, we have $[A, B] \in \mathfrak{g}$.

**Proof.** First we claim that for any $C \in G$, we have $CBC^{-1} \in \mathfrak{g}$. One can show (using the implicit function theorem, for instance) that for some $\epsilon > 0$, there exists a path $\beta : (\epsilon, \epsilon) \to G$ such that $\beta'(0) = B$. Since $t \mapsto C\beta(t)C^{-1}$ is a path in $G$ sending $0 \mapsto G$, its derivative at 0, which is $CBC^{-1}$, lies in $\mathfrak{g}$.

Now there exists $\delta > 0$ and a path $\alpha : (-\delta, \delta) \to G$ such that $\alpha(0) = A$. A similar calculation to the one in Exercise 2.1.2 shows that

$$\frac{d}{dt} \left( \alpha(t)B\alpha(t)^{-1} - B \right) \bigg|_{t=0} = [A, B].$$

Since $t \mapsto \alpha(t)B\alpha(t)^{-1} - B$ is a path in $\mathfrak{g}$, its derivative at 0 lies in $\mathfrak{g}$ as well.

\[\square\]

**2.3** The construction above is simple in the case $G = \text{SL}_n(\mathbb{R})$. The derivative of $\text{det} : \text{GL}_n(\mathbb{R}) \to \mathbb{R}$ at $I$ is $\text{tr} : \mathfrak{sl}_n(\mathbb{R}) \to \mathbb{R}$, so

$$\mathfrak{sl}_n(\mathbb{R}) := \{A \in \mathfrak{gl}_n(\mathbb{R}) \mid \text{tr} A = 0\}$$

is the Lie algebra of $\text{SL}_n(\mathbb{R})$. In this case Proposition 2.2.1 is obvious, since $\text{tr}(AB) = \text{tr}(BA)$ and hence $\text{tr}([A, B]) = 0$ for any $A, B \in \mathfrak{gl}_n(\mathbb{R})$.

For $G = \text{O}_n(\mathbb{R})$, observe that

$$\mathfrak{o}_n(\mathbb{R}) = \{A \in \mathfrak{gl}_n(\mathbb{R}) \mid A^T = -A\}$$

is the Lie subalgebra of skew-symmetric matrices. Indeed, differentiating the relation $A^T A = I$ yields $A^T + A = 0$. Since $\text{SO}_n(\mathbb{R})$ is the connected component of $\mathfrak{o}_n(\mathbb{R})$ containing the identity matrix, we have

$$\mathfrak{so}_n(\mathbb{R}) = \mathfrak{sl}_n(\mathbb{R}) \cap \mathfrak{o}_n(\mathbb{R}) = \mathfrak{o}_n(\mathbb{R}).$$

Let $G = \text{SO}_2(\mathbb{R})$ be the circle group. According to the previous paragraph $\mathfrak{so}_2(\mathbb{R})$ consists of those $2 \times 2$ matrices $A$ such that $A^T = -A$. Such matrices have the form

$$\begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$$

for $a \in \mathbb{R}$, and in particular they form a line in $\mathfrak{gl}_2(\mathbb{R}) \cong \mathbb{R}^4$. Since $\text{SO}_2(\mathbb{R})$ is abelian, so is its Lie algebra, meaning the Lie bracket on $\mathfrak{so}_2(\mathbb{R})$ vanishes identically.

**Exercise 2.3.1.** Show that a one-dimensional Lie algebra is abelian.

We now consider the rotation group $G = \text{SO}_3(\mathbb{R})$ in three dimensions and explain the structure of its Lie algebra $\mathfrak{so}_3(\mathbb{R}) = \mathfrak{o}_3(\mathbb{R})$ in more familiar terms. As the angle $\theta$ varies, the matrices $R_\theta^x, R_\theta^y, R_\theta^z$ defined above make up three subgroups isomorphic to the circle group $\text{SO}_2(\mathbb{R})$. In particular, they form smooth paths through $I$, which can be differentiated to obtain tangent vectors

$$r^x := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad r^y := \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad r^z := \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \mathfrak{so}_3(\mathbb{R}).$$
These matrices form a basis of the three-dimensional Lie algebra \( \mathfrak{so}_3(\mathbb{R}) \), i.e. the linear map

\[
\mathbb{R}^3 \longrightarrow \mathfrak{so}_3(\mathbb{R})
\]

which sends \((a, b, c) \mapsto ax + by + cz\) is an isomorphism. Since the commutator bracket is bilinear and skew-symmetric, it is determined by the three equations

\[
[r^x, r^y] = r^z, \quad [r^y, r^z] = r^x, \quad [r^z, r^x] = r^y.
\]

**Exercise 2.3.2.** Verify these relations among \(r^x, r^y, r^z\).

Recall that \(\mathbb{R}^3\) has a skew-symmetric bilinear product \((v, w) \mapsto v \times w\), called the **cross product**, uniquely characterized by

\[
e_1 \times e_2 = e_3, \quad e_2 \times e_3 = e_1, \quad e_3 \times e_1 = e_2.
\]

Here \(e_1, e_2, e_3\) is the standard basis of \(\mathbb{R}^3\), sometimes denoted by \(i, j, k\). Geometrically \(v \times w\) is a vector of length \(|v||w|\) perpendicular to both \(v\) and \(w\), with its orientation determined by the “right-hand rule”: as the fingers on your right hand sweep out the arc from \(v\) to \(w\), your thumb points in the direction of \(v \times w\).

The isomorphism (2.3.1) sends \(e_1 \mapsto r^x, e_2 \mapsto r^y\), and \(e_3 \mapsto r^z\), so the calculations above show that the cross product in \(\mathbb{R}^3\) corresponds to the commutator bracket in \(\mathfrak{so}_3(\mathbb{R})\).

**2.4 The Heisenberg Lie algebra** is significant in both classical and quantum mechanics. As a vector space \(\mathfrak{g} = \mathbb{R}^{2n+1}\) with the coordinates labeled as \(q_1, \cdots, q_n, p_1, \cdots, p_n, z\). The Lie bracket is determined by the **canonical commutation relations**

\[
[q_i, q_j] = 0, \quad [p_i, p_j] = 0, \quad [q_i, p_j] = 0, \quad [q_i, p_i] = z, \quad [z, q_i] = 0, \quad [z, p_i] = 0, \quad \text{for } i \neq j.
\]  

(2.4.1)

where \(i \neq j\). We remark that \(\mathfrak{g}\) can also be realized as the subalgebra of \(\mathfrak{gl}_{n+2}(\mathbb{R})\) consisting of matrices whose only nonzero entries lie in the first row and last column, with the \((1, 1)\) and \((n+2, n+2)\) entries equal to zero. For example, the 5-dimensional Heisenberg algebra can be viewed as the space of matrices of the form

\[
\begin{pmatrix}
0 & q_1 & q_2 & z \\
0 & 0 & 0 & p_1 \\
0 & 0 & 0 & p_2 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

**Exercise 2.4.1.** Show that the commutator bracket on matrices of this form satisfies the relations (2.4.1), whence this matrix Lie algebra is isomorphic to the Heisenberg Lie algebra.

The Heisenberg algebra is contained within the following infinite-dimensional Lie algebra, which will appear later as the Poisson bracket of observables in Hamiltonian mechanics. Let \(C^\infty(\mathbb{R}^{2n})\) be the space of smooth real-valued functions on \(\mathbb{R}^{2n}\). We label the coordinates \(q_1, \cdots, q_n, p_1, \cdots, p_n\). The **Poisson bracket** of \(f, g \in C^\infty(\mathbb{R}^{2n})\) is defined by the formula

\[
\{f, g\} := \sum_{i=1}^n \left( \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right).
\]

**Exercise 2.4.2.** Check that the Poisson bracket makes \(C^\infty(\mathbb{R}^{2n})\) into a Lie algebra, and that the Leibniz identity

\[
\{f, gh\} = \{f, g\} h + g \{f, h\}
\]

holds for \(f, g, h \in C^\infty(\mathbb{R}^{2n})\).

We remark that the Lie subalgebra of \(C^\infty(\mathbb{R}^{2n})\) spanned by the coordinates \(q_1, \cdots, q_n, p_1, \cdots p_n\) and the constant function 1 is the \((2n + 1)\)-dimensional Heisenberg algebra, with 1 corresponding to \(z\). Indeed, it is easy to check that the relations (2.4.1) are satisfied.

Here is another example of an infinite-dimensional Lie algebra occurring in geometry. The vector space \(C^\infty(\mathbb{R}^n, \mathbb{R}^n)\) of smooth functions \(\mathbb{R}^n \to \mathbb{R}^n\) can be interpreted as the space of smooth vector fields on \(\mathbb{R}^n\).
For $1 \leq i \leq n$, we denote by $\frac{\partial}{\partial x_i}$ the constant vector field with value $e_i \in \mathbb{R}^n$. A general vector field $\xi$ on $\mathbb{R}^n$ can be written

$$\xi = f_1 \frac{\partial}{\partial x_1} + \cdots + f_n \frac{\partial}{\partial x_n}$$

for some smooth functions $f_1, \cdots, f_n : \mathbb{R}^n \to \mathbb{R}$. Vector fields act on functions by the obvious formula, suggested by the notation:

$$\xi \cdot f := f_1 \frac{\partial f}{\partial x_1} + \cdots + f_n \frac{\partial f}{\partial x_n}.$$ 

The *Lie bracket* of vector fields $\xi = \sum_i f_i \frac{\partial}{\partial x_i}$ and $\zeta = \sum_i g_i \frac{\partial}{\partial x_i}$ on $\mathbb{R}^n$ is defined by the formula

$$[\xi, \zeta] := \sum_{i=1}^n (\xi \cdot g_i - \zeta \cdot f_i) \frac{\partial}{\partial x_i}$$

$$= \sum_{i=1}^n \sum_{j=1}^n \left( f_j \frac{\partial g_i}{\partial x_j} - g_j \frac{\partial f_i}{\partial x_j} \right) \frac{\partial}{\partial x_i}.$$

**Exercise 2.4.3.** Show that $C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ is a Lie algebra under the Lie bracket of vector fields, and that the Leibniz identity

$$[\xi, f \zeta] = (\xi \cdot f) \zeta + f [\xi, \zeta]$$

holds for all $\xi, \zeta \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ and all $f \in C^\infty(\mathbb{R}^n)$. 