**Associated primes**

**Motivation:**

Let \( n \) be an integer.

We can write its unique prime factorization as

\[
    n = \pm p_1^{d_1} \cdots p_s^{d_s}.
\]

In fact, in \( \mathbb{Z} \), \((n) = (p_1^{d_1}) \cap \cdots \cap (p_s^{d_s}).\)

We will see that the "associated primes" of \((n)\) are the \((p_i)\) and the primary components of \((n)\) are the \((p_i^{d_i})\).

We will use these concepts to generalize the unique factorization of integers to arbitrary rings.

**Geometric motivation:**

Let \( R = k[x_1, \ldots, x_n], \) and \( I \subseteq R \) an ideal.

**Def:** The closed set \( V(I) \) is reducible if it can be written \( V(I) = V(I') \cup V(I'') \) where \( V(I) \) is not equal to \( V(I') \) or \( V(I'') \).

Otherwise \( V(I) \) is irreducible.

**Claim:** \( V(I) \) is irreducible \( \Leftrightarrow \sqrt{I} \) is prime.
Pf: If $\sqrt{I}$ is prime, then if $V(I) = V(\sqrt{I}) = V(I') \cup V(I'')$ and $\sqrt{I} \in V(I')$, then $V(I') \supseteq V(\sqrt{I})$, so they're equal.

If $\sqrt{I}$ isn't prime, $fg \in \sqrt{I}$ for $f, g \notin \sqrt{I}$.

So for $P \in V(\sqrt{I})$, $f \in P$ or $g \in P$.

$\Rightarrow V(\sqrt{I}) = V(\sqrt{I}, f) \cup V(\sqrt{I}, g)$

Neither of which is equal to $V(\sqrt{I})$ since $f, g \notin \sqrt{I} = \bigcap_{P \ni \sqrt{I}} P$

We'll see that $\sqrt{I}$ can be written in a unique minimal way as a finite intersection of primes. This is the "primary decomposition" of $\sqrt{I}$ and corresponds to writing $V(I)$ in the unique minimal way as the union of irreducible closed sets.

Ex: Define $I := (x^2, xy) \subseteq k[x, y]$.

Geometrically, this is $V(x^2) \cap V(xy)$

which is, geometrically, roughly the line $x = 0$ w/ additional "scheme" structure (i.e. a tangent direction) at the origin.
We will see, purely algebraically, that the associated primes are $(x)$ and $(x, y)$.

However, we can write $I = (x) \cap (x^2, y)$ or $I = (x) \cap (x^2, xy, y^3)$

radical $= (x, y)$  \hspace{2cm} radical $= (x, y)$

So the description as the intersection of ideals whose radicals are the associated primes is not unique.

More precisely, let $R$ be a ring and $M$ an $R$-module.

**Def:** A prime $P$ of $R$ is associated to $M$ if there is some $x \in M$ s.t. $P = \text{ann}_R(x) = \{ r \in R \mid rx = 0 \}$.

The set of all primes associated to $M$ is denoted $\text{Ass}_R M$, or just $\text{Ass} M$ if the ring is clear.

(sometimes the associated primes of $R/I$ over $R$ are just called the associated primes of $I$.)

**Remark:** If $P \in \text{Ass} M$, then $P = \text{ann}_R(x)$, so

$$R \xrightarrow{x} M \text{ has kernel } P, \text{ so } R/P \cong \text{ a submodule of } M.$$  

Conversely, if $P$ is some prime ideal s.t. $R/P \hookrightarrow M$ as modules, then $P$ is the annihilator of the image of $1$. That is:
\[ P \text{ is an associated prime of } M \implies R/P \text{ is isomorphic to a submodule of } M. \]

Now we state some important results about associated primes.

**Theorem:** Let \( R \) be a Noetherian ring and \( M \neq 0 \) a finitely generated \( R \)-module. Then

a.) \( \text{Ass } M \) is finite and nonempty, each containing \( \text{ann}(M) \).
   It includes all primes minimal among those containing \( \text{ann } M \).

b.) \[ \bigcup_{P \in \text{Ass } M} P = \{ \text{zero-divisors on } M \} \cup \{0\} \]

c.) \( \text{Ass } M \) commutes with localization, i.e. if \( U \in R \) is multiplicatively closed, then
   \[ \text{Ass}_{R[U^{-1}]} M[U^{-1}] = \{ PR[U^{-1}] \mid P \in \text{Ass } M \text{ and } P \cap U = \emptyset \}. \]

We'll prove this in the next section after a few lemmas.

**Remark:** Why can we find primes minimal over an ideal? Let \( \{Q_i\} \) be a chain of prime ideals containing \( I \).

Then if \( a, b \in \bigcap Q_i \), \( a \) or \( b \) is in all \( Q_i \), so \( \bigcap Q_i \) is prime.

That is, every chain has a lower bound, so Zorn's Lemma...
implies that there exist minimal primes over $I$.

(Noe that this holds for even non-Noetherian rings!).

**Def:** The primes in $\text{Ass } M$ that are not minimal are called **embedded** primes of $M$.

If $M = \frac{R}{I}$, then if $P$ is an embedded prime in $R$, $V(P)$ is called an **embedded component** of $\text{Spec } (\frac{R}{I})$.

If $P$ is a minimal associated prime, $V(P)$ is an **isolated component** of $\text{Spec } (\frac{R}{I})$.

**Ex:** Let's go back to the example of $I = (x^2, xy) \subseteq R$.

What is $\text{Ass } (\frac{R}{I})$? The only nonzero elements annihilated are multiples of $x$ or $y$.

$\text{ann } (x) = (x, y)$ and $\text{ann } (y) = (x)$. $\Rightarrow \text{Ass } (\frac{R}{I}) = \{(x), (x, y)\}$