Rational functions

Let $\mathcal{V} \subseteq \mathbb{A}^n$ be a variety (i.e. irreducible).

$\Gamma(\mathcal{V})$ is an integral domain $\Rightarrow$ we have the following def:

**Def:** The field of rational functions on $\mathcal{V}$, denoted $k(\mathcal{V})$, is the field of fractions of $\Gamma(\mathcal{V})$. $f \in k(\mathcal{V})$ is a rational function on $\mathcal{V}$.

**Ex:** In $\mathcal{V}(xy - z^2) \subseteq \mathbb{A}^3$, $\frac{x}{z}$ is the same rational function as $\frac{z}{y}$.

**Def:** A rational function $f \in k(\mathcal{V})$ is defined or regular at $P \in \mathcal{V}$ if $\exists g, h \in \Gamma(\mathcal{V})$ s.t. $f = \frac{g}{h}$ and $h(P) \neq 0$.

In the previous example, $f = \frac{x}{z} = \frac{z}{y}$ is defined at $(x, y, z)$ if $z \neq 0$ or $y \neq 0$.

**Poles of rational functions**

**Def:** Let $f \in k(\mathcal{V})$, $P \in \mathcal{V}$. $P$ is a pole of $f$ if $f$ is not defined at $f.$ (i.e. every possible denominator vanishes at $P$).

**Ex:** 1) If $\Gamma(\mathcal{V})$ is a UFD, then up to multiplication by units, $f \in k(\mathcal{V})$ can be written uniquely as $f = \frac{a}{b}$, where $a, b$ rel. prime, so the pole set of $f$ is $\mathcal{V}(b)$. 
2.) In the $V(xy-z^2)$ example, the pole set of $\frac{x}{z} = \frac{z}{y}$ is $V(\overline{z}, \overline{y}) \subseteq V(xy-z^2)$.

Prop.: The set of poles of a rational function is an algebraic subset of $V$.

Pf.: Suppose $V \subseteq A^n$. Let $f \in k(V)$. Let $J_f = \{ g \in \Gamma(V) \mid gf \in \Gamma(V) \}$

Easy to check: $J_f$ is an ideal.

WTS: $V(J_f) =$ pole set of $f$.

P is not a pole of $f \iff \exists a, b \in \Gamma(V)$ s.t. $\frac{a}{b} = f$, $b(P) \neq 0$

$\iff \exists b \in J_f$ s.t. $b(P) \neq 0$.

$\iff P \notin V(J_f)$. □

Local rings at points

Def.: Let $P \in V$. $\mathcal{O}_P(V) \subseteq k(V)$ is the set of rational functions on $V$ that are defined at $P$, called the local ring of $V$ at $P$.

Caution: $\mathcal{O}_P(V) \neq k(P)$.

Ex.: $P = V(x) \subseteq A'$
\[ \Gamma(P) = \frac{k[x]}{(x)} \cong k \text{ so } k(P) \cong k. \]

However, \( \frac{x}{1} \in \mathcal{O}_p(A') \), but \( \frac{1}{x} \notin \mathcal{O}_p(A') \), so \( \mathcal{O}_p(A') \) is not a field.

Although \( x \) evaluated at \( P \) is \( 0 \), \( x \neq 0 \) in \( \mathcal{O}_p(A') \).

More generally, \( \mathcal{O}_p \) depends on \( V \), whereas \( k(P) = \Gamma(P) \cong k \) always.

Claim: \( \mathcal{O}_p(V) \) is a subring of \( k(V) \).

Pf: \( \frac{a}{b}, \frac{c}{d} \in \mathcal{O}_p(V) \) s.t. \( b(P), d(P) \neq 0 \). Then \( b(P)d(P) \neq 0 \), so products and differences are in \( \mathcal{O}_p(V) \).

So \( k \subseteq \Gamma(V) \subseteq \mathcal{O}_p(V) \subseteq k(V) \) \( \square \)

Prop: \( \Gamma(V) = \bigcap_{p \in V} \mathcal{O}_p(V) \), for \( V \) a variety. (i.e. \( \Gamma(V) \) is exactly the rational functions defined at every point of \( V \))

Pf: We know \( \subseteq \). If \( f \in \bigcap_{p \in V} \mathcal{O}_p(V) \), then \( f \) has no poles, so if \( J_f = \{ g \in \Gamma(V) \mid gf \in \Gamma(V) \} \), \( V(J_f) = \emptyset \).

\( \Rightarrow 1 \in J_f \) (Weak Nullstellensatz) \( \Rightarrow f \in \Gamma(V) \). \( \square \)

If \( f \in \mathcal{O}_p(V) \), we can evaluate at \( P \):
If \( f = \frac{a}{b} = \frac{a'}{b'} \) \( \Rightarrow \) \( a(P)b'(P) = a'(P)b(P) \Rightarrow \frac{a(P)}{b(P)} = \frac{a'(P)}{b'(P)}. \)  

i.e. evaluation of \( f \) is well-defined.

Evaluation gives us a homomorphism:

\[
\mathcal{O}_P(V) \rightarrow k  \\
f \rightarrow f(P)
\]

Since \( k \in \mathcal{O}_P(V) \) maps to itself, this map is surjective.

The kernel is thus max', called the maximal ideal of \( V \) at \( P \), defined \( \mathfrak{m}_P(V) = \{ \text{non-units of } \mathcal{O}_P(V) \} = \{ f | g \in \mathfrak{I}_r(P) \} \)

**Def/Lemma:** A ring \( R \) is a local ring if it satisfies the following equivalent conditions:

1.) The set of non-units in \( R \) is an ideal.

2.) \( R \) has a unique maximal ideal \( \mathfrak{m} \).

(i.e. \( \mathcal{O}_P(V) \) is, in fact, a local ring)

**Pf:** 1.) \( \Rightarrow \) 2.): let \( m \) be the ideal consisting of non-units. If \( \mathfrak{I} \not= R \) is an ideal, it contains no units, so \( \mathfrak{I} \subseteq m \).

2.\) \( \Rightarrow \) 1.): let \( m \) be the unique maximal ideal. Then if
\( a \in R \) is not a unit, \((a) \not\subseteq R\), so \((a) \subseteq m\). Thus \(m\) contains all non-units, so it is exactly the set of non-units. \( \square \)

**Ex:** 1.) Let \( R = \{ \frac{a}{b} \mid a, b \in \mathbb{Z}, \ b \text{ odd}\} \).

\( R \) is a ring (check)

\( c \in R \) is a non-unit \( \iff c = \frac{2a}{b} \) where \( b \) is odd

\( \iff c \in (2) \)

Thus, \( R \) is a local ring.

2.) \( \mathbb{C}[x] \) is not a local ring: \( x+1 \) and \( x \) are non-units, but \( (x+1) - x = 1 \), so the non-units don't form an ideal.

3.) Let \( R = \{ \frac{a}{b} \in k(x) \mid a, b \in k(x) \text{ and } b \text{ has a nonzero constant term}\} \)

**Exer:** \( R \) is a local ring w/ max' ideal \( (\frac{x}{1}) \).

**In fact:** \( R=\mathcal{O}_p(A') \).

**Prop:** \( \mathcal{O}_p(V) \) is Noetherian.

**Pf:** Let \( I \subseteq \mathcal{O}_p(V) \). WTS \( I \) is finitely generated.

Consider \( J = I \cap \Gamma(V) \). \( J \) is an ideal of \( \Gamma(V) \).
\[ \Gamma(V) \text{ is } \text{Noetherian, so } J = (f_1, \ldots, f_r) \subseteq \Gamma(V). \]

Let \( f \in \mathfrak{I} \subseteq \mathcal{O}_p(V) \). Then \( f = \frac{a}{b} \), \( a, b \in \Gamma(V) \), \( b(p) \neq 0 \).

Thus, \( bf = a \in \mathfrak{I} \cap \Gamma(V) = J \).

\[ \Rightarrow bf = a_1 f_1 + \ldots + a_r f_r, \quad a_i \in \Gamma(V). \]

\[ \Rightarrow f = \left( \frac{a_1}{b} \right) f_1 + \ldots + \left( \frac{a_r}{b} \right) f_r \Rightarrow \mathfrak{I} = (f_1, \ldots, f_r). \square \]

Let \( \mathcal{Y} : V \rightarrow W \) be a regular map of affine varieties.

Consider \( \mathcal{Y}^* : \Gamma(W) \rightarrow \Gamma(V) \)

\[ \begin{array}{ccc}
\mathcal{K} & \mathcal{K} \\
\mathcal{K}(W) & \mathcal{K}(V) \\
\downarrow & \downarrow \\
\mathcal{Y}^* \end{array} \]

\( \square \) Can we extend \( \mathcal{Y}^* \) to \( \mathcal{K}(W) \)?

If so, there's only one possible map:

\[ \begin{array}{ccc}
g & \mapsto & \mathcal{Y}^*(g) \\
h & \mapsto & \mathcal{Y}^*(h) \\
\mathcal{Y}^* \end{array} \]

But if \( \ker \mathcal{Y}^* \), this doesn't work!

(Note: it does work if \( \mathcal{Y} \) is dominant! Do you see why this is true geometrically?)

Instead, let \( P \in V \). Set \( Q = \mathcal{Y}(P) \).

Let \( h \in \Gamma(W) \) s.t. \( h(Q) \neq 0 \).
Then \[ V \to W \xrightarrow{h} k \]
\[ P \to Q \to h(Q) \neq 0 \]
\[ \gamma^*(h) \]

Thus, if \[ \frac{g}{h} \] is defined at \( Q \) and \( h(Q) \neq 0 \),

then \[ \frac{\gamma^*(g)}{\gamma^*(h)} \] is defined at \( P \).

In fact, this gives us a well-defined (check!) map, so \( \gamma^* \) induces a morphism

\[ O_Q(W) \to O_P(V). \]

Furthermore, if \[ \frac{g}{h} \in m_Q(W) \], then \( g(Q) = 0 \),

so \[ \gamma^*(g)(P) = g(\gamma(P)) = g(Q) = 0 \]

so \[ \gamma^*(\frac{g}{h}) \in m_P(V) \]. That is, \( m_Q(W) \) gets mapped into \( m_P(V) \).