Affine varieties and coordinate rings

**Def:** An affine variety is an irreducible affine algebraic set.

So \( \{ \text{affine varieties} \} \rightarrow \{ \text{prime ideals} \} \) in \( \mathbb{A}^n \).

**Functions on varieties**

Let \( V \subseteq \mathbb{A}^n \) be a variety. Let \( \mathcal{F}(V, k) \) be the set of all functions \( V \rightarrow k \). \( \mathcal{F}(V, k) \) has a natural ring structure.

**Def:** \( f \in \mathcal{F}(V, k) \) is a polynomial function or regular function on \( V \) if there is some \( F \in k[x_1, \ldots, x_n] \) s.t. \( f = F|_V \).

i.e. \( \forall (a_1, \ldots, a_n) \in V, \ F(a_1, \ldots, a_n) = f(a_1, \ldots, a_n) \).

**Easy exercise:** The set of regular functions on \( V \) is a subring of \( \mathcal{F}(V, k) \).

**Def:** The ring of regular functions on \( V \) is called the coordinate ring of \( V \). It's denoted \( \Gamma(V) \).

**Ex:** 1.) \( \Gamma(\mathbb{A}^n) = k[x_1, \ldots, x_n] \).

2.) Let \( V = V(y - x^2) = \{(t, t^2) \mid t \in k\} \).
The function \( y \) outputs the \( y \)-coordinate, i.e. it is projection onto the \( y \)-axis. The function \( x^2 \) is the same function on \( V \).

3.) Consider \( V(xy - 1) \subseteq \mathbb{A}^2 \). Is \( \frac{1}{y} \) regular?

\[
xy = 1 \Rightarrow x = \frac{1}{y} \quad \text{so} \quad x \text{ and } \frac{1}{y} \text{ are the same function on } V(xy - 1), \text{ so } \frac{1}{y} \text{ is regular.}
\]

In general, if \( V \subseteq \mathbb{A}^n \) is a variety, we have a restriction map

\[
k[\mathbb{A}^n] \rightarrow \Gamma(V)
\]

whose kernel is precisely the functions that vanish on \( V \), i.e. \( I(V) \). So we have...

**Prop/Def:** \( \Gamma(V) \cong k[\mathbb{A}^n]/I(V) \)

**Remark:** \( \Gamma(V) \) is ring-finite over \( k \), and if \( V \) is a variety, since \( I(V) \) is prime, \( \Gamma(V) \) is an integral domain.

**Def:** A subvariety of \( V \) is a variety \( W \subseteq \mathbb{A}^k \) s.t. \( W \subseteq V \).
Thus, \[ \{ \text{subvarieties of } V \} \leftrightarrow \{ \text{prime ideals} \} \text{ in } \Gamma(V) \]
\[ \{ \text{points of } V \} \leftrightarrow \{ \text{maximal ideals} \} \text{ in } \Gamma(V) \]

We can define the function \( \Gamma(V) \rightarrow \Gamma(W) \) to be the restriction map \( \bar{f} \mapsto f|_W \), where \( f \in k[x_1, \ldots, x_n]/I(V) \), \( f \cdot k[x_1, \ldots, x_n] = R \)

\( \bar{f} \) is in the kernel \( \iff \bar{f} \) vanishes on \( W \iff \bar{f} \in I_V(W) \)

So \[ \Gamma(W) \cong \frac{\Gamma(V)}{I_V(W)} \cong \frac{(R/I(V))}{I(V)} \]

\textbf{Ex.:} Going back to \( V = V(xy - 1) \subseteq \mathbb{A}^2 \)

\begin{center}
\begin{tikzpicture}
\end{tikzpicture}
\end{center}

As we saw, \( y = \frac{1}{x} \) in \( \Gamma(V) \), so
\[ \Gamma(V) = \frac{k[x,y]}{(xy - 1)} \cong k[x, \frac{1}{x}] \], i.e.

\textit{Laurent polynomials.}