**Algebraic sets**

Let \( k \) be a field. For our purposes, we only want to consider fields that are algebraically closed. (We will see why soon.)

**Def:** A field \( k \) is algebraically closed if every polynomial with coefficients in \( k \) has a root in \( k \). That is, if \( f \in \mathbb{k}[x] \), there is some \( a \in k \) s.t. \( f(a) = 0 \).

**Note:** By induction, this implies that every polynomial in \( \mathbb{k}[x] \) is the product of linear factors. i.e \( f = \beta(x-a_1)...(x-a_n) \) for some \( \beta, a_1, ..., a_n \in k \).

**Ex:** \( \mathbb{C} \) is algebraically closed (see Math 123), but \( x^2 + 1 \) has no roots in \( \mathbb{IR} \) so \( \mathbb{IR} \) is not algebraically closed.

**Def:** Affine \( n \)-space, denoted \( \mathbb{A}_k^n \), or just \( \mathbb{A}^n \), is the set of \( n \)-tuples of elements of \( k \) (to distinguish it from the vector space \( k^n \)).

Let \( f \in \mathbb{k}[x_1, ..., x_n] \). Then \((a_1, ..., a_n) \in \mathbb{A}^n_k \) is a zero of \( f \) if \( f(a_1, ..., a_n) = 0 \).

**Ex:** Conics in \( \mathbb{A}^2_k \).
A conic is the set of zeros of a quadratic equation:

\[ g(x,y) = ax^2 + bxy + cy^2 + dx + ey + f \]

If we just consider the real locus, we get the following familiar conics:

- ellipse
- parabola
- hyperbola
- two lines (e.g. \( xy \))

If \( g \) is reducible over \( \mathbb{R} \)

But the \( \mathbb{R} \) locus doesn't always give a very complete picture:

In \( \mathbb{R}^2 \): \( x^2 + y^2 \) defines a single point, and \( x^2 + y^2 + 1 \) defines the empty set, whereas they have infinitely many solutions over \( \mathbb{C} \).

**Note:** It usually makes sense to require \( a, b, \) or \( c \) to be nonzero to avoid lines (e.g. \( x \)) and the whole plane (0).

More generally, we can describe zero loci using more than
Def: Let \( S \subseteq k[x_1, \ldots, x_n] \) be a set of polynomials. Define

\[
V(S) := \{ P \in \mathbb{A}^n \mid f(P) = 0 \text{ for all } f \in S \}
\]

\( X \subseteq \mathbb{A}^n \) is an algebraic set if \( X = V(S) \) for some \( S \).

IOM

Let \( S \subseteq k[x_1, \ldots, x_n] \), and let \( I \) be the ideal generated by \( S \).

Claim: \( V(S) = V(I) \).

Pf: If \( P \in V(I) \), then \( f(P) = 0 \) \( \forall f \in S \) since \( S \subseteq I \). Thus, \( P \in V(S) \).

If \( P \in V(S) \), then if \( g \in I \), \( g = a_1 f_1 + \ldots + a_m f_m \) for some \( a_i \in k[x_1, \ldots, x_n] \), \( f_i \in S \). Thus \( g(P) = 0 + \ldots + 0 = 0 \), so \( P \in V(I) \). \( \square \)

Cor: Every algebraic set is equal to \( V(I) \) for some ideal \( I \).

Ex: If \( f \in k[x_1, \ldots, x_n] \), and \( I = (f) \), then \( V(f) = V(I) \). In this case, \( V(I) \) is called a hypersurface.
If \( f = y \in k[x,y], \) \( V(f) = \) The \( x\)-axis.

We can now deduce several basic properties of \( V(S) \):

1.) It's inclusion-reversing: If \( I \subseteq J \) then \( V(I) \supseteq V(J) \). 
   
   \[ \text{Ex: } (x) \subseteq (x,y) \] 
   
   \[ V(x) \subseteq V(x,y) \]

2.) If \( \{I_x\} \) is a collection of ideals,
   
   \[ \bigcap V(I_x) = V(\bigcup I_x) = V(\text{ideal gen. by } I_x) \]
   
   (i.e. intersections of algebraic sets are alg. sets.)

3.) \( f, g \in k[x_1, \ldots, x_n] \Rightarrow V(f) \cup V(g) = V(fg) \).

   More generally, \( V(I) \cup V(J) = V(IJ) \)
   
   (i.e. finite unions of alg. sets are alg. sets).

[Note: Infinite (even countable!) unions are not always alg. sets: If \( X = V(I) \subseteq \mathbb{A}^n \), then \( f \in I \) is 0 or has finitely many roots, so \( X \) is finite or all of \( \mathbb{A}^n \).]
4.) $V(0) = \mathbb{A}^n$, $V(1) = \emptyset$, and $V(x_1 - a_1, \ldots, x_n - a_n) = \{(a_1, \ldots, a_n)\}$
so any finite set is algebraic.

Properties 2.) - 4.) show that algebraic sets behave like closed sets. In fact...

**Def:** $X \subseteq \mathbb{A}^n$ is a Zariski closed set if $X$ is an algebraic set. $Y$ is Zariski open if $\mathbb{A}^n \setminus Y$ is Zariski closed.

The collection of Zariski open sets in $\mathbb{A}^n$ is called the Zariski topology.

**Ex:** The Zariski open sets on $\mathbb{A}^1$ are the empty set and the cofinite sets.

**Note:** If you know any topology, the Zariski topology is strictly coarser than the standard Euclidean topology. i.e. algebraic sets are all closed in the standard topology as well.