Math 137 - Problem Set 6  
Due Wednesday, Mar 11

All rings are commutative, and $k$ is an algebraically closed field.

1. Let $k = \mathbb{C}$. Find the multiple points, and the tangent lines at the multiple points, for each of the following curves:
   
   (a) $y^3 - y^2 + x^3 - x^2 + 3xy^2 + 3x^2y + 2xy$
   (b) $x^4 + y^4 - x^2y^2$
   (c) $x^3 + y^3 - 3x^2 - 3y^2 + 3xy + 1$

2. The ring of formal power series over $k$ is defined
   
   $$ k[[x]] := \left\{ \sum_{i=0}^{\infty} a_i x^i \mid a_i \in k \right\}. $$

   Show that $k[[x]]$ is a DVR with uniformizing parameter $x$.

3. If a curve $f$ of degree $n$ has a point $P$ of multiplicity $n$, show that $f$ consists of $n$ lines through $P$ (not necessarily distinct).

4. Let $T : \mathbb{A}^2 \to \mathbb{A}^2$ be a regular map, and $T(Q) = P$.
   
   (a) If $f \in k[x, y]$, show that the following inequality of multiplicities holds:
   
   $$ m_Q(T^*(f)) \geq m_P(f). $$

   (b) (Extra credit) Let $T = (T_1, T_2)$, and define
   
   $$ J_QT = \begin{pmatrix} \frac{\partial T_1}{\partial x}(Q) & \frac{\partial T_1}{\partial y}(Q) \\ \frac{\partial T_2}{\partial x}(Q) & \frac{\partial T_2}{\partial y}(Q) \end{pmatrix} $$

   to be the Jacobian matrix of $T$ at $Q$. Show that if $J_QT$ is invertible, $m_Q(T^*(f)) = m_P(f)$. Does the converse hold?

You might want to wait until after class on Monday to work on the next two problems.

5. Let $P = (0, 0)$ and $k = \mathbb{C}$. Consider the following affine plane curves containing $P$:
   
   - $A = x^2 - y$
   - $B = y^2 - x^3 + x$
   - $C = y^2 - x^3$
   - $D = y^2 - x^3 - x^2$
   - $E = (x^2 + y^2)^3 - 4x^2y^2$
(a) Compute $I_P(A, C)$.
(b) Compute $I_P(C, D)$.
(c) Compute $I_P(B, E)$.

6. Let $f$, $g$, and $h$ be affine plane curves.
   
   (a) If $P$ is a simple point on $f$, show
   
   $$I_P(f, g + h) \geq \min\{I_P(f, g), I_P(f, h)\}.$$  
   
   (b) Give an example to show that (a) may be false if $P$ is not simple on $f$.

**Bonus questions** (Extra credit)

7. Let $P = (0, 0)$ lie on an irreducible curve $f$. Let $m = m_P(f)$ be the maximal ideal of $\mathcal{O}_P(f)$.

   (a) Show that $\dim_k(m^n/m^{n+1}) = n + 1$ for $0 \leq n < m_P(f)$.
   
   (b) Conclude that $P$ is a simple point if and only if $\dim_k(m/m^2) = 1$; otherwise $\dim_k(m/m^2) = 2$.

8. Let $X$ and $Y$ be affine varieties, and $\phi : X \to Y$ a regular map. We showed on the last homework that $X$ and $Y$ are locally ringed spaces, with structure sheaves $\mathcal{O}_X$ and $\mathcal{O}_Y$, respectively. Look up the definition of a morphism between locally ringed spaces (e.g. here), and show that $\phi$ is a morphism of locally ringed spaces.