Math 137 - Problem Set 2
Due Wednesday, Feb 12

All rings are commutative, and \( k \) is an algebraically closed field.

1. Let \( I \) be an ideal in a ring \( R \). Show that there is a one-to-one correspondence between radical ideals in \( R \) containing \( I \) and radical ideals in \( R/I \). (Convince yourself that the same holds for prime ideals. You don’t need to write up that part.)

2. (a) Let \( I \subset k[x_1, \ldots, x_n] \) be an ideal. Show that \( I \) is radical if and only if it is equal to the intersection of all the maximal ideals containing it.
   
   (b) Show that the radical of the ideal \( I = (x^2 - 2xy^4 + y^6, y^3 - y) \subset \mathbb{C}[x, y] \) is the intersection of three maximal ideals.

3. Let \( X = V(x^2 - yz, xz - x) \subset \mathbb{A}^3_{\mathbb{C}} \). Find the irreducible components of \( X \) and their corresponding prime ideals. Make sure you justify your solution.

4. Let \( a_1, a_2, \ldots, a_n \in k \). Show that \( (x_1 - a_1, \ldots, x_n - a_n) \subset k[x_1, \ldots, x_n] \) is a maximal ideal. (Hint: reduce to the case where the \( a_i \) are all 0.)

5. Let \( X \subset \mathbb{A}^n \) be a set (not necessarily algebraic). The Zariski closure of \( X \), denoted \( \overline{X} \), is the intersection of all Zariski closed sets containing \( X \). Show that \( V(I(X)) = \overline{X} \).

6. A subset of affine space \( U \subset \mathbb{A}^n \) is called compact (in the Zariski topology) if for every collection \( \{U_i\}_{i \in J} \) (where \( J \) is some indexing set) of Zariski open sets such that
   
   \[
   U \subset \bigcup_{i \in J} U_i,
   \]

   then \( U \) is also contained in some finite union of the \( U_i \). That is, there is some finite set \( L \subset J \) such that
   
   \[
   U \subset \bigcup_{i \in L} U_i.
   \]

   More concisely, \( U \) is compact if every open cover has a finite subcover. Show that if \( X \subset \mathbb{A}^n_k \) is an algebraic set, \( X \) is compact in the Zariski topology.

**Bonus topology questions** (Extra credit)

7. Identify \( \mathbb{A}^1 \times \mathbb{A}^1 \) with \( \mathbb{A}^2 \) in the natural way. Show that the Zariski topology on \( \mathbb{A}^2 \) is not the product topology induced by the Zariski topology on \( \mathbb{A}^1 \).

8. For each \( f \in k[x_1, \ldots, x_n] \) define \( U_f \) to be the set of points \( P \in \mathbb{A}^n \) such that \( f(P) \neq 0 \). Prove that the collection of all such sets \( U_f \) forms a basis for the Zariski topology on \( \mathbb{A}^n \).