Recall: If a group $G$ acts on a nonempty set $A$, then for every $g \in G$, the function

$$\sigma_g : A \to A \text{ defined } \sigma_g(a) = g \cdot a$$

is a bijection, and the function

$$\phi : G \to S_A \text{ defined } \phi(g) = \sigma_g$$

is a homomorphism. The kernel of $\phi$ is called the kernel of the action, and the action is faithful if the kernel is the identity.

**Note:** The kernel of the action $K$ is a normal subgroup and we can give $G/K$ an action on $A$ as follows:

$$gK \cdot a = g \cdot a.$$

Check this is well-defined: If $gK = hK$ then $g = h k$ for some $k \in K$.

$$\Rightarrow g \cdot a = h k \cdot a = h \cdot (k \cdot a) = h \cdot a$$

acts as identity

Then $G/K \cong \text{im}(\phi) \leq S_A$, so it has trivial kernel, so it's faithful!

**Group actions from maps to $S_A$**

We can also get group actions from morphisms to $S_A$, i.e. if $A$ is a set and $G$ a group s.t.
\( \varphi: G \to S_A \) is a homomorphism, define the group action of \( G \) on \( A \) as follows:

For \( g \in G \), \( a \in A \), \( g \cdot a = \varphi(g)(a) \)

This is in fact an action (axioms are easy to check), and all actions of \( G \) on \( A \) arise in this way. i.e.

**Prop:** There is a bijection between the actions of \( G \) on \( A \) and the homomorphisms \( G \to A \).

**Def:** If \( G \) is a group, a permutation representation of \( G \) is any homomorphism \( G \to S_A \), for some nonempty set \( A \); thus given an action of \( G \) on \( A \).

**Orbits and stabilizers**

Let \( G \) be a group acting on a set \( A \). Recall from a HW problem that we can define an equivalence relation on \( A \) as follows:

\[ a \sim b \iff a = g \cdot b, \text{ some } g \in G. \]

We already showed this is an equivalence relation, and the equivalence classes are called orbits. For \( a \in A \), the equivalence class containing \( a \) is called the orbit of \( a \).
Note that the stabilizer of $a$, $Ga := \{g \in G \mid g \cdot a = a\}$ gives us no additional elements of the orbit of $a$, and in fact, each coset of $Ga$ gives us an additional element of the orbit of $a$. That is,

**Prop:** For $a \in A$, the number of elements in the orbit of $a$ is $|G: Ga|$. 

**Pf:** We show that there is a bijection between the cosets of $Ga$ and the elts of the orbit of $a$. Call the orbit $O_a$.

We define a map $O_a \rightarrow \text{cosets of } Ga$ by

$$b = g \cdot a \mapsto g \cdot Ga.$$ 

This is well-defined: if $b = g \cdot a = h \cdot a$, then $h^{-1}g \cdot a = a \Rightarrow h^{-1}g \in Ga$. Thus, $(h^{-1}g)Ga = 1Ga \Rightarrow gGa = hGa$.

This is clearly surjective, and it’s injective since if $gGa = hGa$, then $h^{-1}g \in Ga \Rightarrow h^{-1}g \cdot a = a \Rightarrow g \cdot a = h \cdot a$. $\square$

**Def:** The action is transitive if there’s only one orbit, i.e. if

$\forall a, b \in A \exists g \in G \text{ s.t. } a = g \cdot b$.

**Ex:** $S_n$ always acts transitively on $\{1, \ldots, \ell\}$: if $a, b \in \{1, \ldots, \ell\}$,

$$(a \cdot b) \cdot a = b.$$