\( K(x) = 1 \)

\( X \) is a minimal model of \( k = 1 \), then

\[ \gamma = \varphi \mid_{dK_x} : x \to C \] is an elliptic fibration

**Step 1** classify fibers

**Step 2** canonical bundle formula

**Theorem** \( \alpha : X \to C \) be a relatively minimal elliptic fibration, with multiple fibers \( m_1 F_1, \ldots, m_n F_n \)

\[
\omega_X = \alpha^* \left( \omega_C \otimes \left( R_{\alpha_x}^! \Theta_x \right)^v \right) \otimes \Theta \left( \sum (m_i - 1) F_i \right) \]

\[
K_X \sim_{Q} \alpha^* \left( K_C + L + B \right)
\]

\[
B = \sum \frac{m_i - 1}{m_i} \varphi_i \quad m_i F_i = \alpha^* (p_i)
\]

\[ \deg \left( R_{\alpha_x}^! \Theta_x \right)^v = \chi (\Theta_x) \]
**PF Sketch**

**Lemma 1:**

1) \( \alpha^{-1}(p) = mF \), then

\[
\mathcal{O}_F = \mathcal{O}_F \quad \mathcal{O}_F(F) \text{ is a torsion bundle of order } m
\]

2) \( h^0(\mathcal{O}_{\alpha^{-1}(p)}) = h^1(\mathcal{O}_{\alpha^{-1}(p)}) = 1 \)

for all \( p \in C \)

\[
h^0(F, \mathcal{O}_x(K_x)|_F) = h^0(F, \omega_F) = h^1(F, \mathcal{O}_F) \uparrow \text{adjunction}
\]

For all fibers

**Grauert's thm \(\Rightarrow\) \( \alpha_x^* \mathcal{O}_x(K_x) \) is a line bundle

\[
\alpha_x^* \alpha_x^* \mathcal{O}_x(K_x) \cong \mathcal{O}_x(K_x)
\]
Claim 1 \( \alpha \) is an isomorphism on the generic fiber \( X_m \) of \( \alpha \)

\[ \mathcal{U} \subseteq X \text{ locus where } \alpha \text{ is smooth} \]

\[ \omega_u = \omega\big|_u = (\alpha^*\omega_c \otimes \omega_{x/c}) \big|_u = \alpha^*\omega_c \otimes \omega_u \big|_{\alpha(u)} \]

restrict to \( X_m \)

\[ \omega_{X_m} = \Theta_{X_m} \]

\[ \Rightarrow \quad \Theta_x(K_x) = \alpha^*\Theta_x(K_x) \otimes \Theta_x(D) \]

\( D \) effective and contained in fibers

\[ 0 = k_x^2 = (\alpha^*B + D)^2 = D^2 \]

\[ \Rightarrow \quad D = \sum n_i \alpha^{-1}(p_i) \quad \text{for some} \ n \]

\[ \Theta_{F_i} = \omega_{F_i} = \Theta_x(k_x + F_i) \big|_{F_i} = \Theta_x(n_iF_i + F_i) \big|_{F_i} \]

\[ \Theta_{n_i+1} = m_i \quad n_i = m_i - 1 \]
Next \[ \alpha_{\ast} \Theta_{x} (K_{x}) = \alpha_{\ast}(\alpha_{\ast} w_{c} \otimes w_{y_{c}}) \]

[Relative Duality]

\[ \alpha_{\ast} w_{x_{c}} = (R_{\alpha_{\ast}} \Theta_{x}) \]

Use \( R \Theta \) to compute \( \) Leray spectral \( \) sequence

\[ X(\Theta_{x}) = - \deg R_{\alpha_{\ast}} \Theta_{x} \]

\[ k = 2 \]

\( X \) is a minimal model of general type

by Abundance, we know

\[ \phi = \phi_{x} : X \to \mathbb{Z} \quad \dim \mathbb{Z} = 2 \]

\[ \text{Proj } R(K_{x}) \]

\[ E \leq Exc(\phi) \]

\[ E^{2} < 0 \]

\[ K_{x} \cdot E = 0 \]
\( k_E = (K_X + E) \cdot E < 0 \Rightarrow E \cong \mathbb{P}^1 \)

\( E^2 = -2 \)

\[ \Rightarrow \quad \text{Exc}(\varphi) = \text{disjoint union of trees of (-2) curves with negative definite intersection matrix} \]

Classified by Dynkin diagrams of type ADE

\[ \Rightarrow \quad \mathbb{Z} \text{ has ADE singularities at points where some (-2) curves are contracted} \]

\[ \phi^* \quad \text{Pic}(\mathbb{Z}) \xrightarrow{\varphi} \text{Pic}(X) \xrightarrow{\oplus} \mathbb{Z} \quad \text{E} \in \text{Exc} \]

Compute this using \( \text{Pic} = H^1(X, \mathcal{O}_X^*) \)

\( + \quad \varphi^* \mathcal{O}_X = \mathcal{O}_Z \)

\[ \begin{align*}
K_X &= \varphi^* L \\
L &\in \text{Pic}(\mathbb{Z}) \\
K_Z &= \varphi^* K_Z \\
\Rightarrow \quad L &= K_Z
\end{align*} \]

\( L \mid_{\mathbb{Z}^\text{sm}} = K_Z \mid_{\mathbb{Z}^\text{sm}} \)

\( K_X = \varphi^* K_Z \quad \text{so} \quad K_Z \text{ is ample} \)
if $C \not\in \text{Exc}(\Psi)$ on $X$ \quad \uparrow
\Rightarrow K_X \cdot C > 0
\Rightarrow K_Z \cdot C' > 0 \text{ for any } C \in Z$

$\varphi : X^{\min} \rightarrow Z = X^\text{can} = \text{Proj} \ R(K_X)$

uniquely characterized by
1) how big canonical sing
2) $K_Z$ is ample

\underline{Goal}

1) Cone + contraction Theorems
   + generalize to singularities

2) Base point Free theorem
   rationality, + Vanishing theorems
   + non-Vanishing

The cone of curves in higher dimension

$X$ projective
$D$ cartier divisor, $C \in X$ a curve
$D \cdot C = \deg i^* O_X (D)$

$C = \sum a_i C_i$ \quad 1-cycle, extend by linearity
$C \equiv C'$ iff $D.C = D'.C'$ for all Cartier $D$.

$N_1(x) := N_1(x)_R = \text{numerical eq classes of IR-1-cycle } a_i \in \mathbb{R}$.

$N^+(x) = NS(x) \otimes \mathbb{R}$

$NS(x) = \text{Pic}(x)/_{\text{Pic}^0(x)} = \text{divisors up to alg eq}$

$N_1(x) \otimes N^1(x) \rightarrow \mathbb{R}$ perfect pairing.

$P(x) = \dim N_1(x) < \infty$ for any $x$.

$x$ smooth, $N_1(x) \rightarrow H_2(x, \mathbb{R})$.

$NE(\mathbb{Q})(x) = \{ \Sigma a_i [C_i] \mid 0 \leq a_i \in \mathbb{Q} \}$

$NE(x) = \{ \sum a_i [C_i] \mid 0 \leq a_i \in \mathbb{R} \}$

$\overline{NE}(x) = \text{closure inside } \leq N_1(x)$.

$D_{\text{Cartier}} := \{ x \in N_1(x) \mid D.x \geq 0 \}$.
\[ \overline{\text{NE}}(x) := \overline{\text{NE}}(x) \cap D_{>0} \]

**Thm (Kleiman's criterion)** If \( x \) is projective, \( D \) cartier, then \( D \) is ample iff

\[ D_{>0} \supseteq \overline{\text{NE}}(x) \setminus \{0\} \]

**Cor 1)** \( \overline{\text{NE}}(x) \) doesn't contain a line

2) \( \exists z \mid H \cdot z \leq c^2 \) for any fixed \( c \) is compact

3) the set of integral 1-cycles \( \overline{\text{NE}}(x) \cap \mathbb{Z} = \sum n_i [c_i] \) w/ \( H \cdot z \leq c^2 \) is finite
**Thm. (Cone Theorem I)**

Let $X$ be a smooth projective variety. Then there exist countably many rational curves $C_i$ such that

1) $\overline{NE}(X) = \overline{NE}(X)_{K_X \geq 0} + \sum R_{\geq 0} [C_i]$

where $0 < -(K_X, C_i) \leq \dim X + 1$

2) For any ample $H$, $\geq 0$

$$\overline{NE}(X) = \overline{NE}(X)(K_X + \epsilon H) \geq 0 \text{ finite}$$
**Rmk**  \( R = \mathbb{R} \geq 0 [c_i] \) are extremal rays

meaning if \( n_w \in \text{NE}(X) \)

s.t. \( n + w \in R \implies n_w \in R \)

**Def**  \( F \subseteq \text{NE}(X) \) any extremal face \( (K_X - \text{negative}) \)

a morphism \( \text{cont}_F : X \rightarrow Y \) is

an extremal contraction of \( F \) if

1) \( \text{cont}_F (c) = p \text{t} \quad \Leftrightarrow \quad [c] \in F \)

2) \( \text{cont}_F \mathcal{O}_X = \mathcal{O}_Y \)

**Thm (Contraction Theorem I)**

Let \( R \) is a \( K_X \)-negative extremal ray then there is unique extremal contraction to a projective \( Y \)

\( \text{cont}_R : X \rightarrow Y \), and

if \( L \) a line bundle on \( X \) w/ \( L \cdot c = 0 \) for all \( [c] \in R \)

then \( \exists L^p \) on \( Y \) s.t. \( \text{cont}_R^* L^p = L \) line bundle