Minimal Models in dim 2

**Def** \( X \) s.t. \( K_X \) is nef

**Thm** (Nonvanishing)

If \( X \) is a minimal model, then \( H^0(X, mK_X) \neq 0 \) for some \( m > 0 \)

**If** suppose we had a minimal model \( X \) but w/ \( \beta(X) = -\infty \)

\[ \Rightarrow \alpha : X \rightarrow E \overset{\text{elliptic curve}}{\leftarrow} \]

\( \alpha \) is smooth & \( g(F) \geq 1 \)

**Thm** \( \alpha' : X \rightarrow E \) smooth proper w/ fibers \( g(F) \geq 1 \), \( g(E) \leq 1 \)
then \( \alpha \) is isotrivial.

\[ g \geq 2 \]

\[ E' \times F \cong X' \xrightarrow{t'} X \quad \text{where } t \]
\[ \alpha' \downarrow \quad \downarrow \alpha \quad \text{is } \text{étale} \]

\[ E' \xrightarrow{t} E \]
$g = 1$

\[ \alpha^* \omega_{x/E} \] this is a torsion line bundle

\[ \sim \rightarrow t : E' \rightarrow E \]

\[ t \text{ \textit{etale}} \]

**PF sketch**

\[ g > 2 \]

holomorphic

| bounded |

\[ C, P' = \tilde{E} \rightarrow H_g \]

| lift |

holomorphic

complex analytic quotient

Since $H_g$ is bounded, this lift is constant

\[ \Rightarrow \chi = \tilde{E} \times \tilde{E} \times \tilde{E} \rightarrow \tilde{E} \] is a product

but $\text{Aut}(F)$ finite $\Rightarrow$ factor through a finite cover $E'$

\[ g = 1 \]

\[ E \rightarrow \tilde{E} \]

\[ \text{"} \tilde{M}_{g,1} = \text{space of genus 1 curve with a point} \]
X \xrightarrow{\alpha} E \quad \text{might not have a section}

\begin{align*}
\text{&}
\text{&} \quad J(x) \\
\text{&} \quad J_{\alpha}(x) \rightarrow E
\end{align*}

Start of a beautiful story about hyperbolicity of moduli.

Positivity of $\alpha_{*} W_{y/E}$

Iitaka's conjecture on subadditivity of Kodaira dimensions

Then (Iitaka)

Suppose $h : X' \rightarrow X$ finite unramified

\Rightarrow \quad K(X') = K(X)

\begin{align*}
\text{g} \geq 2 \\
K(X) = K(X') = K(F \times E') \geq 1
\end{align*}

\begin{align*}
g(F) &\geq 2 \\
g(E') &\geq 1
\end{align*}

\text{Contradiction!}

\begin{align*}
g=1 \\
X' \xrightarrow{\alpha} X \\
E' \xrightarrow{\alpha} E
\end{align*}
\[
\alpha^* \omega_{X/E} = \alpha^! \omega_{X/E} = \mathcal{O}_E
\]

\[
\alpha^! \omega_X = \alpha^! (\alpha^* \omega_{E'} \otimes \omega_{X/E'})
\]

\[
\cong \omega_{E'} \otimes \alpha^! \omega_{X/E'} = \omega_{E'}
\]

Projection formula \[ g(E') \geq 1 \]

\[
\Rightarrow H^0(X, \alpha^* (K_X)) = H^0(E', \omega_{E'}) = g(E') \geq 1
\]

\[
\Rightarrow K(X) = K(X') \geq 0
\]

\begin{center}
Contradiction!
\end{center}

\[ \text{§2: Abundance in dim 2} \]

\[ \text{Thm: If } X \text{ is a minimal model in dim 2, then } K_x \text{ is semiample} \]

\[ mK_X \text{ is bpf} \]

\[ K_x \text{ nef } \Rightarrow K_x \text{ semiample} \]
Proof by nonvanishing $K(x) > 0$

$K^2 = 2$ (general type)

It's a corollary of the full base point free theorem

Thm $X$ smooth proj variety

$D = \sum d_i D_i$, $0 < d_i < 1$

$r$ singularities at rational $D$ are "nice" $(X, D)$ has KLT singularities

If $L$ nef cartier divisor such that $\alpha L - (K_X + D)$ is ample for some $\alpha > 0$

$\Rightarrow mL bpf $ for some $m >> 0$

$K = 1$

$K_X^2 = 0$, $K_X \neq 0$ (numerical Kodaira dim)

nef but not big
\[ |mK_x| = |M| + F \quad \text{fixed part} \]
\[ \emptyset \quad \text{moving part of } mK_x \quad M \text{ nef} \]

\[ 0 \leq M^2 \leq M, (M+F) \leq (M+F)^2 = (mK_x)^2 = 0 \]
\[ \implies M^2 = M.F = F^2 = 0 \]

moving divisor \( M \) \( \text{w} \) \( M^2 = 0 \)
\[ \implies |M| \text{ is base point free} \]
\[ \text{b/c } \bigcap M' = \emptyset \]
\[ M' \in |M| \]

\[ \emptyset : X \longrightarrow \mathbb{Z} \leftarrow \text{curve} \]

\( F \) satisfying

1) \( F.M = 0 \)
\[ \implies F \in \text{fiber of } \emptyset \]

2) \( F^2 = 0 \)
\[ \implies F = \sum m_i F_i \]

\( F_i \) are fibers of \( \emptyset \)
\[ mK_x = M + F = \varphi_{lm1}^* H + \sum_{\pi_i \in \mathcal{E}} m_i \varphi^{* \pi_i} \]
\[ = \varphi^{* (H + \sum_{\pi_i \in \mathcal{E}} m_i \pi_i)} \]

Pull back of ample by a morphism is by \( f \)

\[
\begin{align*}
K &= 0 \\
h^0(X, \mathcal{O}(mK_x)) &\leq 1 \quad \text{for any } m > 0 \\
\chi(\mathcal{O}_x) &\geq 0 \quad \text{for geometric genus} \\
1 - h^1(\mathcal{O}_x) + h^0(\mathcal{O}_x(K_x)) &\geq 0
\end{align*}
\]

\( \begin{cases} \text{Cases} \end{cases} \)

i) \( p_g = q = 0 \) \hspace{1cm} 2k \sim 0 \quad \text{Enriques}

ii) \( p_g = 0, \; q = 1 \) \hspace{1cm} mK_x \sim 0 \quad \text{bielliptic}

iii) \( p_g = 1, \; q = 0 \) \hspace{1cm} K_x \sim 0 \quad \text{K3 surface doesn't exist}

iv) \( p_g = q = 1 \) \hspace{1cm} mK_x \sim 0 \quad \text{k3 surface}

v) \( p_g = 1, \; q = 2 \) \hspace{1cm} \text{abelian surface}
Need to check that $mK_x \sim 0$ for some $m \geq 0$

1) $P_g = q = 0$, $\chi(\mathcal{O}_x) = 1$

**Castelnuovo's Rationality**

if $P_2 = H^0(X, \mathcal{O}_x(2K_x)) = 0$

$\Rightarrow$ x rational, can't happen

$\Rightarrow$ $H^0(X, \mathcal{O}_x(2K_x)) \neq 0$

want $h^0(X, \mathcal{O}_x(-2K_x)) \geq 1$

$\Rightarrow 2K_x \sim 0$

$h^0(X, \mathcal{O}_x(-2K_x)) + h^2(X, \mathcal{O}_x(-2K_x))$

$\equiv \frac{1}{2}(-2K_x)(-2K_x - K_x) + \chi(\mathcal{O}_x)$

$= h^0(\mathcal{O}_x(3K_x)) = 1$

$\Rightarrow h^0(X, \mathcal{O}_x(3K_x)) = 0$

$E_x$ if $h^0(X, \mathcal{O}_x(2K_x)) = 1 = h^0(X, \mathcal{O}_x(3K_x))$

$\Rightarrow h^0(X, \mathcal{O}_x(K_x)) = 1$

$\Rightarrow h^2(X, \mathcal{O}_x(3K_x)) = 0$
\(\quad p_g = 0 \quad q_v = 1 \quad x(C_\infty) = 0\)

\(\alpha': X \to E \approx \text{elliptic curve}\)

Run the same argument as in non vanishing proof

\[ x \approx \text{smooth fibration} \]

\[ g(F) \geq 1 \]

\[ g \geq 2 \]

\[ X^1 = E' \times F \to X \]

\[ g(F) = 2 \]

Can't happen

\[ g = 1 \]

\[ \omega_x = \alpha^* \omega_E \otimes \omega_{x/E} = \alpha^*(R^1\alpha_*\mathcal{O}_x) \]

\[ \alpha^* \omega_{x/E} \cong (R^1\alpha_*\mathcal{O}_x)^\vee \]

Relative duality

But \(\alpha^* \omega_{x/E}\)

Torsion bundle on \(E\)

\[ \Rightarrow \quad (\alpha^* \omega_{x/E})^\otimes m = \mathcal{O}_E \]

\[ \Rightarrow \quad \omega_{x} \otimes m = \alpha^*(\alpha_* \omega_{x/E})^\otimes m = \alpha^* \mathcal{O} = \mathcal{O}_{x} \quad m > 1 \quad \text{w/} \quad p_3 = 0 \]
iii) \( p_g = 1 \quad g = 0 \quad \chi(\mathcal{O}_x) = 2 \)

\[
h^0 (\mathcal{O}_x (-K_x)) + h^0 (\mathcal{O}_x (2K_x))
\geq 0 + \chi(\mathcal{O}_x) = 2
\]

\( h^0 (\mathcal{O}_x (2K_x)) \leq 1 \quad \text{by} \quad K(x) = 0 \)

\( \implies h^0 (\mathcal{O}_x (-K_x)) \geq 1 \)

\( \implies K_x \sim 0 \)

iv) \( p_g = g = 1 \quad \chi(\mathcal{O}_x) = 1 \)

\[ h'(x, \mathcal{O}_x) = \dim \text{Pic}(x) \]

\[ \text{Pic}^0 (x) = \frac{H^1 (x, \mathcal{O}_x)}{-\text{im} (H^1 (x, \mathbb{Z}))} \neq 0 \]

\[ 0 \to \mathbb{Z} \to \mathcal{O}_x \to \mathcal{O}_x^+ \to 1 \]

Pick some \( \mathcal{M} \in \text{Pic}^0 (x) \)

2-torsion but not trivial

\[ h^0 (\mathcal{O}_x (\mathcal{M})) = 0 \]

\[ h^2 (\mathcal{O}_x (\mathcal{M})) = h^0 (\mathcal{O}_x (K_x - \mathcal{M})) \]

\[ \geq \frac{1}{2} \chi(K_x - \mathcal{M}) + \chi(\mathcal{O}_x) = 1 \]
\[ G \in |K_x - M| \Rightarrow 2G \in |2K_x - 2M| = |2K_x| \]

\[ D \in |K_x| \]

\[ h^0(C_0(2K_x)) \leq 1 \quad |2K_x| \triangleright 2D = 2G \]

\[ K_x \sim D = G \sim K_x - M \quad \Rightarrow M \sim O \quad \text{contradiction} \]

v) \( P_g = 1 \quad q = 2 \quad \chi(C_0) = 0 \)

\[ \alpha: X \to \text{Alb}(X) = A \quad \dim A = 2 \quad \text{abelian surface} \]

Claim 1

\( \alpha \) does not factor through a curve \( \alpha \)

\[ X \to C \to A = \text{Jac}(C) \]

\[ g(C) = 2 \]
Pick some \( t' : C' \to C \) etale cover with \( g(C') \geq 2 \)
\[
x \to C
\]
\[
x'^{1} \to C' \to \text{Jac}(C') = \text{Alb}(x')
\]

\[
K(x') = K(x) = 0 \quad \text{dim} \geq 3
\]
\[
q(x') \geq 3 \quad \text{not possible}
\]

so \( \alpha \) is surjective

\underline{Riemann-Hurwitz} \hspace{1cm} \leftrightarrow \hspace{1cm} \text{effective}

\[
K_x = \alpha^* K_A + E
\]

Use intersection products

+ that \( t \) component of \( E \) or \( \text{deg} \) int

to conclude \( E \) contains a \( g(C') = 1 \)

\( \implies \text{have an elliptic curve} \)

\( \text{on } A \)

Complete reducibility theorem