Surfaces with $K_X$ nef

**Def**: A minimal model is a smooth projective surface $X$ with $K_X$ nef.

$\Rightarrow$ minimal surfaces

**Thm** (Nonvanishing)

Suppose $K_X$ is nef, then for some $m > 0$,

$H^0(X, \mathcal{O}_X(mK_X)) \neq 0$

**Cor** if $K_X$ nef, then $\chi(X) > 0$

**Cor** (Coarse classification)

Every surface $X$ is birational to a minimal surface $X_m$ where $X_m$ is a minimal model.
\[ \langle \Rightarrow \rangle \; \chi(x) \geq 0 \]

. \( X_m \) is ruled or \( \mathbb{P}^2 \)

\[ \iff \; h^0(X) = + \infty \]

\underline{Thus} (Abundance)

If \( K_x \) is nef, then \( K_x \) semiample

\[ |mK_x| \text{ bpf for some } m \]

\underline{Rmk} \; \text{Nonvanishing + abundance}

\[ \Rightarrow \; \text{Classification of surfaces by Kodaira dimension} \]

Essentially need to classify to prove these theorems

§2: Background

Big divisor, Iitaka dimension

\[ L \text{ is a Cartier divisor on } X \]

\[ K(X, L) := \max \left\{ \dim \mathfrak{m}_{L}^{i} (x)^{2} \leq \dim (x) \right\} \]
$N(L) := \{ m \mid (m-1) \neq 0 \}$ numerical semigroup of \( L \)

$$K(x, K_x) = K(x)$$

\[ \text{Prop} \] \( L \) has Hirata dim \( K \)

\[ \iff \] there exist \( a, A \) s.t.

for all \( m \in N(L) \) large

\[ a m^k \leq h^0(x, \mathcal{O}_X(mL)) \leq A m^k \]

\[ \text{Def} \] \( L \) is big if \( h^0(x, L) = \dim X \)

\[ \iff \]

\[ h^0(x, \mathcal{O}_X(mL)) \geq C_m \dim X \]

for \( m \in N(L) \)

\[ \text{Kodaira's Lemma} \]

Suppose \( D \) is big \& \( E \) effective

then \( h^0(x, \mathcal{O}_X(mD - E)) \neq 0 \)

for \( m \in N(L) \) large
Cor TFAE
1) \( D \) is big
2) \( \exists \) an ample \( A \) s.t.
   \[ D = A + N \quad N \text{ effective} \]

Cor Suppose \( D \) is nef then

\[ D \text{ is big} \iff D^\dim_x > 0 \]

Cor \( X \) is a minimal model in \( \dim 2 \)

\( K_X \) is nef

then \( K(X) = 2 \iff K_X^2 > 0 \)

Albanese morphism

Thm let \( X \) smooth projective

then \( \exists \) abelian variety \( \text{Alb}(X) \)

\( k \) a morphism

\[ \alpha : X \to \text{Alb}(X) \]

1) if \( \beta : X \to T \) with \( T \) abelian,
3) \[ \mathfrak{F} \xrightarrow{B} \mathfrak{T} \xrightarrow{\pi} \mathfrak{A} \mathfrak{lb}(x) \]

2) \( \mathfrak{p}^\#: H^0(\mathfrak{A} \mathfrak{lb}(x), \mathfrak{S}_\mathfrak{lb}(x)) \rightarrow H^0(x, \mathfrak{S}_x^1) \)

is an isomorphism.

\[ \dim \mathfrak{A} \mathfrak{lb}(x) = \dim H^0(x, \mathfrak{S}_x^1) = h_1 = q(x) \leftarrow \text{irregularity} = h_0 \]

3) \( \mathfrak{A} \mathfrak{lb}(x) \) is generated by \( \mathfrak{a}(x) \)

4) If \( \xrightarrow{f} \) \( x \rightarrow y \) \( f \) is surjective

\[ \Rightarrow \mathfrak{a}(f) \) surjective \]

\[ \mathfrak{A} \mathfrak{lb}(x) \rightarrow \mathfrak{A} \mathfrak{lb}(y) \]

5) \( \mathfrak{a}: X \rightarrow C = \mathfrak{a}(x) \rightarrow \mathfrak{A} \mathfrak{lb}(x) \)

if \( \mathfrak{a} \) factors through a smooth curve \( \mathfrak{C} \), then \( \mathfrak{A} \mathfrak{lb}(x) = \mathfrak{J} \mathfrak{a} \mathfrak{c} \mathfrak{c} (\mathfrak{C}) \)

\[ \mathfrak{g}(\mathfrak{C}) = h^0(x, \mathfrak{S}_x^1) = q(x) \]
Sketch of construction

\[ T = \frac{V}{\Gamma} \]

\( V \) \( \mathbb{C} \)-vector space
\( \Gamma \) integer lattice
torus complex

Polarization - to make it projective

Fact any \( u : T_1 \to T_2 \)

is up to translation is induced by

\( a : V_1 \to V_2 \) linear map

s.t. \( a(\Gamma_1) \subseteq \Gamma_2 \)

\[ \Rightarrow u^* : H^0(T_2, \mathcal{O}_{-1}^\perp) \to H^0(T_1, \mathcal{O}_{-1}^\perp) \]

\[ V_2^* \xrightarrow{a^*} V_1^* \]

\( A_{\text{Mo}}(x) = H^0(x, \mathcal{O}_x^\perp)^* \)

\( H = \text{im} (H_1(x, \mathbb{Z}) \to H^0(x, \mathcal{O}_x^\perp)^*) \)

Polarization comes from Hodge theory
8 \longrightarrow (w \mapsto \int w \in \mathcal{C})

\alpha : X \rightarrow \text{Alb}(X)

\alpha(x) = (w \mapsto \int w)

\zeta_x \circ \zeta_x^* \in H_1(X, \mathbb{R})

§ 3: Nonvanishing

X is a minimal model in dim 2
then \quad H^0(X, K_X) \neq 0

Lemma
k_X is nef & K(X) \leq 0
then \quad X(\Theta_X) > 0

Proof
k_X is nef but not big so
k_X^2 = 0

Noether's Formula
\begin{align*}
12 \chi(\mathcal{O}_x) &= k^2 + e(x) \\
&= e(x) = b_0 - b_1 + b_2 - b_3 + b_4
\end{align*}

\begin{align*}
\chi(\mathcal{O}_x) &= h^0(\mathcal{O}_x) - h^1(\mathcal{O}_x) + h^2(\mathcal{O}_x) \\
&\leq 2 - h^1(\mathcal{O}_x) \quad \text{by (**)}
\end{align*}

\begin{align*}
b_1 &= h^1(x, \mathcal{O}_x) + h^0(x, \mathcal{O}_x) \\
&= 2 h^1(x, \mathcal{O}_x)
\end{align*}

\begin{align*}
12 \chi(\mathcal{O}_x) &= 2 - 2 h^1(x, \mathcal{O}_x) + b_2 \\
&= 2 - 4 h^1(x, \mathcal{O}_x) + b_2 \\
&\geq 2 + 4 \chi(\mathcal{O}_x) - 8 + b_2 \\
&= -6 + 4 \chi(\mathcal{O}_x) + b_2
\end{align*}
\[ 8 x(\mathcal{O}_x) > -6 + b_2 \geq -6 \]

\[ x(\mathcal{O}_x) \geq -\frac{6}{8} \geq 0 \]

**Proof of Nonvanishing**

Suppose \( K_x \) is negative but \( \kappa(x) = -\infty \)

Then by Lemma above \( x(\mathcal{O}_x) > 0 \)

\[ 1 - h^1(x, \mathcal{O}_x) + h^0(x, \mathcal{O}_x) = 0 \]

Also know \( P_m = h^0(x, \mathcal{O}_x(\text{link}_x)) = 0 \)

in particular \( P_2 = 0 \)

Castelnuovo's criteria: \( P_2 = h^1(x, \mathcal{O}_x) = 0 \)

\[ \Rightarrow x \text{ rational} \]
Can't happen w.l.o.g. \( k_x \) is nef

\[ \Rightarrow h^1(x, O_x) > 0 \Rightarrow h^1(x, O_x) = 1 \]

\[ h^0(x, O_x^1) \]

\( \alpha : X \to \text{Alb}(x) = E \leftarrow \text{elliptic curve} \)

\( \beta \) surjective + connected fibers \( \Rightarrow \text{flat} \)

\( F \leq X \) a general fiber

is a smooth curve \( g(F) = g \)

Claim 1 \( g(F) \geq 1 \)

Since \( k_x \) is nef

\( F^2 = 0 \)

\( k_F = (k_x + F)|_F \geq 0 \)

but \( \deg k_F > 0 \Rightarrow g(F) > 1 \)

Claim 2 \( \alpha \) is smooth i.e. fibers are smooth
Step 1: Suppose \( d^{-1}(p) = F_1 + F_2 \) \( \neq F_2 \) 

Let \( H \) be an ample, \( H, F_1, F_2 \) are linearly independent numerical classes.

\[ \Rightarrow \quad 3 \leq p(x) \leq \dim H^2(x, \mathcal{O}_x) = b_2 \]

\[ 0 = e(x) = 2 - 2h^1(x, \mathcal{O}_x) + b_2 \]

\[ b_2 = 2 \quad \text{Contradiction} \]

\[ \Rightarrow \quad d^{-1}(p) \text{ is irreducible} \]

Step 2: \( d^{-1}(p)_{\text{red}} = F_p \text{ is irreducible} \)

\[ d^{-1}(p) = m_p F_p = F \]

Lemma: \( e(F_p) \geq 2 \chi(\mathcal{O}_{F_p}) \)

With equality \( \iff F_p \text{ smooth} \)

\[ 2 - 2g = 2 \chi(\mathcal{O}_F) \text{ by RR}^{(5p)} \]

smooth case
Use \( n: X^\prime \rightarrow X \) normalization

Compute w/ adjunction:

\[
e(F_p) \geq 2 \times (\Theta_{F_p}) = 2 \frac{1}{m_p} \chi(\sigma_F)
\]

\[
= \frac{1}{m_p} e(F) \geq e(F)
\]

\( g(F) \geq 1 \)

\( \Rightarrow \) \( e(F) \leq 0 \)

\[
o = e(x) = e(F) e(\mathbb{E} \setminus \Delta) + \sum_{p \in \Delta} \epsilon_p
\]

\[
= e(F) e(E) + \sum_{p \in \Delta} (e(F_p) - e(F))
\]

\( E \) elliptic curve

\[
\Rightarrow e(F_p) = e(F) \quad \text{for all } p
\]

\[
= \frac{1}{m_p} e(F)
\]

So either

1) \( e(F) = 0 \) and \( m_p \) arbitrary
2) \( g(F) \geq 2 \) \& \( m_p = 1 \)

\[ \Rightarrow \quad c(F_0) = 2 \chi(\mathcal{O}_{F_0}) \]

so \( F_0 \) smooth

**Thm (Kodaira's Canonical Bundle Formula)**

let \( f : X \to C \) be a minimal genus 1 fibration

\( \chi(F) = 1 \)

\( m_1 F_1, \ldots, m_n F_n \) are nonreduced fibers

\[ \omega_X = f^* \left( \omega_C \otimes (R^1 f_* \mathcal{O}_X) \right) \otimes \mathcal{O}_X \left( \mathcal{O}_X \right) \]

\[ \deg \left( R^1 f_* \mathcal{O}_X \right) = \chi(\mathcal{O}_X) \]

For us, \( \chi(\mathcal{O}_X) = 0 \)

\( C = E \) so \( \omega_E = \mathcal{O}_X \)

\[ \Rightarrow \quad \omega_X = \omega^*(\deg 0 + \deg 70) \]

\( NF_0 = m F \)

ample on a curve

\( \Rightarrow \) has a lot of sections
Contradiction $\Rightarrow \dim \mathcal{M}_g = 1$ even in $g=1$ case so $\alpha$ is smooth.

**Step 3**

$\alpha$ is a smooth family of genus $\geq 1$ curves,
$g(\mathcal{E}) = 1$

**Thm:** If $\alpha: X \rightarrow \mathcal{E}$ is a smooth + proper morphism w/ $g(\mathcal{E}) \leq 1$ & general fiber $F$

If $g(\mathcal{F}) > 2$ then $\alpha$ is isotrivial:

$\mathcal{E} \times \mathcal{F} \cong \mathcal{X} : \alpha \rightarrow \mathcal{E}$

$\mathcal{F} \rightarrow \mathcal{E}$ finite, $\alpha$ étale

Then $g = 1$

$\mathcal{X} \rightarrow \mathcal{E}$ is a torsion line bundle

$\mathcal{E} \rightarrow \mathcal{E}$ finite, $\alpha$ étale

$\mathcal{O}_{\mathcal{E}} = \tau^* \alpha_* \omega_{\mathcal{X}/\mathcal{E}} = \alpha_{*}^! \omega_{\mathcal{X}/\mathcal{E}}$