Inversion of adjunction

Recall $(X, s+\Delta)$ is a log pair

$s$ is integral, normal divisor

there is a unique divisor

\[ \text{Diff}_s (\Delta) \quad \text{s.t.} \]

1) \( (K_X + s + \Delta)|_s \sim_\mathbb{Q} K_s + \text{Diff}_s (\Delta) \)

2) \[ f: (Y, s_Y + \Delta_Y) \to (X, s + \Delta) \]

s.t. \( f_* s_Y = s \) \( f_* \Delta_Y = \Delta \)

\[ f^* (K_X + s + \Delta) = K_Y + s_Y + \Delta_Y \quad \text{crepant} \]

\[ f^* (\Delta_Y|_{s_Y}) = \text{Diff}_s (\Delta) \]

\[ f^* (K_s + \text{Diff}_s (\Delta)) = K_{s_Y} + \Delta_Y|_{s_Y} \]

3) if $\Delta$ is effective, then $\text{Diff}_s (\Delta)$ is effective

4) generalize to the case $s$ is not normal:

\[ u: s^n \to s \quad \text{the normalization} \]

Correct by $K_{s^n/s}$

\[ u: s^n \to \mathbb{P}^2 \]
Prop. Let \((X, \Delta)\) be a plt pair, then TFAE

1) \((X, \Delta)\) plt + 2) \([\Delta]\) normal + 3) \([\Delta]\) is a disjoint union of normal comp

PF: 1) \(\Rightarrow\) 2)

\(f: Y \rightarrow X\) a log resolution

\[ K_Y + \Delta_Y = f^* (K_X + \Delta) + \Delta \]

\[ \Delta_Y = f_*^{-1} \Delta \quad \text{by plt, } \Gamma_{\Delta Y} = E \quad \text{effective } f\text{-exc} \]

\[ S = [\Delta] \quad S_Y = L \Delta_Y \]

\[ f_* S_Y = S \]

\[ 0 \rightarrow \Theta_Y (-s_Y + E) \rightarrow \Theta_Y (E) \rightarrow \Theta_Y (E|_{S_Y}) \rightarrow 0 \]

\[ -S_Y + E = -L \Delta_Y + \Gamma_{\Delta Y} \equiv f_* K_Y + \Theta_Y + \Gamma_{\Delta Y} - \Delta \]

\(f^* (-S_Y + E) \) plt

by GR vanishing, \( R^1 f_* \Theta_Y (-S_Y + E) = 0 \)

\[ \Rightarrow \quad f_* \Theta_Y (E) \rightarrow f_* \Theta_Y (E|_{S_Y}) \rightarrow 0 \]
\[ \mathcal{O}_X \to \mathcal{O}_S \to \mathcal{O}_{S_1} \to f_+ \mathcal{O}_{S_1} (E|_{S_1}) \]

\[ \implies \mathcal{O}_S \to \mathcal{O}_{S_1} \text{ surjective} \]

\[ \implies \text{ isomorphisms } \mathcal{O}_S 
\]

2) \( \implies 3) \) by def

3) \( \implies 1) \) by computing disc

Cor: Suppose \( (X, \Delta) \) is dlt

\( S \in L \Delta \) is component, write \( \Delta = S + \Delta_1 \)

then \( S \) is normal.

**Proof:** \( S + (1-\varepsilon)\Delta_1 + \varepsilon D \) is dlt

Lemma (2.43) \( \not\) not \( \mathbb{Q} \)-currier

\( \exists \ D = \varepsilon \Delta_1 + H \)

ample ?

so \( (X, S + (1-\varepsilon)\Delta_1 + \varepsilon D) \) is dlt

but \( L S + (1-\varepsilon)\Delta_1 + \varepsilon D \) = \( S \)

\( \implies S \) is normal by the previous prop.
**Proof**

Let \((X, s + \Delta)\) be a log pair, \(s\) integral and normal. Then for every divisor \(E \subseteq X\) lying over \(s\), there exists a divisor \(E \subseteq \tilde{X}\) lying over \(X\) such that:

\[
\alpha(E, s, \text{Diff}_s(\Delta)) = \alpha(E, X, s + \Delta)
\]

Moreover,

\[
\text{total disc } (s, \text{Diff}_s(\Delta)) \geq \text{discr}_s (\text{center } s, X, s + \Delta) \\
\geq \text{discr}_s (\text{center } s, X, s + \Delta)
\]

**Proof**

Define \(F: (Y, s_Y + \Delta_Y) \to (X, s + \Delta)\) as a log resolution of the pair.

\[
F^*(K_X + \text{Diff}_s(\Delta)) = K_{s_Y} + (\Delta_Y)_{|s_Y}
\]

Suppose \(s_Y\) is disjoint from \(s\), then all divisors we're dealing with are \(F\)-exc.

Take a sequence of blowups of \(s\) s.t. \(E_s\) appears as a divisor on this blowup, blowup the same sequence of subs of \(s\) but in \(X\), this gives
a divisor $E \geq Y$ s.t. $E|_S = E_Y$ 

s.t. $Center_{S_Y}(E) = E_S_Y$

discrep = - coefficient for a divisor appearing on $Y$

$$(K_Y + S_Y + \Delta_Y)|_S = K_{S_Y} + (\Delta_Y)|_{S_Y}$$

Cor (Easy adjunction)

1) if $(X, s + 0)$ is plt in a nbhd of $S$

$(S, \text{Diff}_S(\Delta))$ is klt

2) if $(X, s + 0)$ is lc in a nbhd of $S$

$(S, \text{Diff}_S(\Delta))$ is lc

Thm (Inversion of adjunction) (Shokurov, Kollár, Kawamata)

The inequalities of discrep above

are $\iff$ therefore the statements

of the cor are $\iff$

MMP with scaling
$(X,0)$ let pair, $H$ be some ample
\[
K_X + D + H \quad \text{nef}
\]
\[
K_X + D + tH = 0 \quad \text{for } t \leq 1
\]
\[
K_X + D = 0
\]
\[
t_1 = \text{nef threshold}
\]
\[
(K_X + D + t,H), R = 0
\]
for some extremal ray $R$

if $t > 0$

$\Phi_R : X \rightarrow Z$ the $(K_X + a)^{-}$ extremal contraction

$(Z, D_2 + t_1 H_2) = \text{wlcM}(X, D + t H / \beta)$

if $\Phi_R$ is small, $X_1 = \text{flip}$

if $\Phi_R$ is divisorial, $X_1 = Z$

if $\Phi_R$ is a Mori fiber space

$(X_1, D_1 + t H_1)$ wlcM \quad \text{so } K_{X_1} + D_1 + t H_1$
\[ t_2 < t_1 \quad \text{and} \quad t_2 = \text{net threshold} \]

\[ \text{LCM}(X_1, \Delta_1 + t_2 H_1 / B) = \text{image of extremal contraction of } R_1 \]

\[ (K_{X_1} + \Delta_1 + t_2 H_1) \cdot R_1 \]

Technical point: need some bigness of \( \Delta \) to guarantee that \( R_1 \) is \((K_{X_1} + q)\)-neg extremal by

\[ X_2 = \text{WLCM}(X_1, \Delta_1 + t_2 H_1 / B) \]

\[ = \text{WLCM}(X_1, \Delta + t_2 H / B) \]

\[ \vdots \]

\[ 0 < \ldots < t_2 < t_1 < 1 \]

Each step of MMP occurs at coefficient \( t_i \), and produces

\[ \text{WLCM}(X, \Delta + t_i H / B) \]

Termination of MMP \( \iff \) Finiteness of \( \text{WLCM} \) for \( (X, \Delta + t H) \) as \( t \in [0, D] \) with scaling
pl-flips (pre-limiting, shokurov)

Def. Let $(X,\Delta)$ plt pair, $\Delta = \Delta s$

$s$ is integral. a pl-flipping contraction of a $(K_x + \Delta)$-flipping contraction

$s$ is $\phi$-ample.

Suppose $Z$ is affine,

existence of $\text{flips} = R(X, K_x + \Delta)$

$\Delta = \Delta s + B$

$B = \Delta s$

$0 \to H^0(X, m(K_x + B)) \to H^0(X, m(K_x + s + B))$

$\to H^0(S, m(K_s + D; \text{Diff} s(B)))$

Res: $R(X, K_x + \Delta) \to R(S, K_s + \text{Diff} s(B))$

Suffices to prove

image of Res is f.g.

$\text{Res}_s(X, K_x + \Delta)$
Existence of minimal models for varieties of log general type

BCHM = Birkar-Cascini-Hacon-McKernan

Thm I \[ f: (x, \Delta) \to B \] \( Q \) -factorial

Let pair projective \( /B \), suppose \( \Delta \)

is \( f \)-big. Then any MMP with scaling terminates.

Thm II \[ f: (x, \Delta) \to B \] as above, suppose either \( \Delta \) is big

and \( K_x + \Delta \) is \( f \)-pseudo effective

OR \( K_x + \Delta \) is \( f \)-big, then

1) \( (x, \Delta) \) has a good LTM \( (x, \Delta / B) \)
2) If $K_X + \Delta$ is $\mathfrak{S}$-big, then $(X, \Delta)$ has a \( \text{LCM}(X, \Delta_B) \)

3) $R_C(T^1, K_X + \Delta) = \bigoplus \delta_x \left( MK_X + \left( \sum \Delta \right) \right)$

is f.g.

**Proof sketch**

if $K_X + \Delta$ is $\mathfrak{S}$-big

$$K_X + \Delta \sim_{\mathfrak{S}, 0} D \geq 0$$

$\Delta' = \Delta + \varepsilon D$, $\Delta'$ is big

$(X, \Delta')$ is klt

$$K_X + \Delta' \equiv (\varepsilon)(K_X + \Delta)$$

$(K_X + \Delta) - \text{mmp} = K_X + \Delta'$ mmp

but now, $\Delta'$ is big so by

then $T$, mmp with scaling terminates

$$X \twoheadrightarrow \text{LTM}(X, \Delta_B)$$

$\triangleright \Leftarrow$ in pseudoeffective
2) in the case $K_x + \Delta$ is big
   is just the bpf

3) In the case $\Delta$ is big

\[ K_x + \Delta = K_x + A + B \quad A \text{ ample} \]
\[ D \hspace{1cm} (x, B) \text{ klt} \]
\[ (K_x + \Delta) - (K_x + B) = \text{ample} \]

In the bpf theorem, $\Rightarrow$ bpf

$\Rightarrow$ models are good + finite generation