**Thm (Cone theorem)**

Let \((X, \Delta)\) be a projective pair with \(\Delta\) effective. Then:

1) There are countably many rational curves \(c_i \in X\) s.t. \(0 < -(K_X + \Delta) \cdot c_i < 2\dim X\) and

\[
\overline{\text{NE}}(X) = \overline{\text{NE}}(X)_{K_X + \Delta \geq 0} + \sum \mathbb{R}_{\geq 0} [c_i]
\]

2) For any \(\varepsilon > 0\) and \(H\) ample

\[
\overline{\text{NE}}(X) = \overline{\text{NE}}(X)_{K_X + \Delta + \varepsilon H \geq 0} \text{ finite}
\]

3) For \(F \subseteq \overline{\text{NE}}(X)\) any \((K_X + \Delta)\)-negative extremal face, \(\exists! \phi_F : X \to \mathbb{P}^1\) which is projective s.t. i) \(\phi_F^* \mathcal{O}_X = \mathcal{O}_{\mathbb{P}^1}\) ii) \(\phi_F(\mathcal{O}) = \mathcal{O}\)

4) If \(L\) is a line bundle on \(X\) s.t. \(L \cdot c = 0\) for \([c] \in F\), then \(L = \phi_F^* L\) for \(L \in \text{Pic}(\mathbb{P}^1)\)
**Proof**

\[ K := K_x + \Delta, \quad a(K_x + \Delta) \]

Recall that denominators in Ratl theorem are bounded by \( a(\dim X + 1) \).

Let \( m \) be a common denominator.

Suppose \( K \) is not nef.

**Step 1**

Let any nef but not ample divisor

\[ 0 \neq F_L = L \cap \overline{\text{NE}(X)}, \quad \text{suppose that} \]

\[ F_L \notin \overline{\text{NE}}_{K \geq 0} \]

\[ \sum_{n, H} = \max_{\ell \in \mathbb{N}} \left\{ nL + H + \frac{tK}{m} \right\} \text{ nef}^2 \]

\[ \sum_{n, H} \in \mathbb{Z} \text{ by rationality} \]
\[ \gamma_{L}(n, H) \leq \gamma_{L}(n+1, H) \]

Pick \( S \in F_{L}^{\text{nef}} \left( nL + H + \frac{\gamma_{L}(n, H)}{m} k \right) \). \( S \geq 0 \)

H. \( \gamma \geq 0 \)

K. \( \gamma \leq 0 \)

C. \( \gamma = 0 \)

\[ \implies \text{the sequence } \gamma_{L}(n, H) \text{ stabilizes to } \gamma_{L}(H) \text{ for } n \geq n_{0}. \]

Take \( n > n_{0} \)

\[ D(n, L, H) = m n L + m H + \gamma_{L}(H) k_{\text{nef}} \]

\[ \Downarrow \]

\[ D(n-1, L, H) \] is also nef

\[ \mid_{F_{D(n, L, H)}} = 0 \implies F_{D(n, L, H)} \subseteq F_{L} \neq 0 \]

\[ F_{D(n, L, H)} + F_{D(n, L, H)} \backslash \delta_{\text{nef}} \subseteq \overline{\text{NE}(x)} \]

\[ \text{b/c } \mid_{F_{D}} = 0, \text{ } H|_{F_{D}} > 0 \implies \mid_{F_{D}} < 0 \]
Step 2. Suppose \( \dim F_L > 1 \), then we can pick \( H \) s.t.

\[
\dim F \left( \bigcap_{i=1}^m (H_i + \sum_k (H_i)_k) \right) < \dim F_L
\]

Pick some ample basis \( H_1, \ldots, H_3 \) for \( N'(x) \)

\[
D(n, L, H_i) \mid_{F_L} = (n H_i + \sum_k (H_i)_k) \mid_{F_L}
\]

the \( H_i \) are linearly independent

so if \( (\text{Span } F_L) \) has \( \dim > 1 \)

\( \implies \) \( D(n, L, H_i) \mid_{F_L} \) can't identically vanish

\( \implies \) \( \exists \) \( D(n, L, H_i) \mid_{F_L} \neq 0 \)

so \( D(n, L, H_i) \subset F_L \neq F_L \)
by repeatedly applying this process we cut down to rays
\[ F_L \text{ nef dim } F_L = 1 \]
\[ F_L \setminus \emptyset \leq \overline{\text{NE}(X)}_{k < 0} \]

each face \( F_L \) for \( L \) nef with \( F_L \not\subseteq \overline{\text{NE}(X)}_{k > 0} \) contains such rays

**Step 3**

\[
\overline{\text{NE}(X)} = \left( \overline{\text{NE}(X)}_{k > 0} + \sum \text{dim } F_L \right) \\
\supseteq \\
\text{by definition}
\]

Suppose that \( w \not\in \overline{\text{NE}(X)} \) then there exists rational \( M \) s.t.
apply rationality theorem to perturb $M$ by some $H + tK$ so that we're on the boundary of $\overline{NE}(x)$ avoiding $W$.

By the previous step, we can cut out a ray of $\overline{NE}(x)$ of the form $F_L$, but not contained in $W$, contradiction.

**Step 4** there are no accumulation points of $F_L$ inside $K < 0$.

\[ F_L = \{ R \in H \}_{m} \]

Pick ample $H_i$ for $L$.

Pick $K, H_1, \ldots, H_{p-1}$ is a basis for $\text{N}^1(x)$.

\[ \frac{H_i \cdot \Xi}{K \cdot \Xi} = \frac{\text{integer}}{m} \quad (\ast) \]

$1 \rightarrow (K, \Xi, H_1, \Xi, \ldots, H_{p-1}, \Xi)$ gives coordinates for $\text{N}^1(x)$.
If we look at \( \mathbb{P}(N^c(x)) \)

\[ U = \{ k < 0 \} \]

\( U \xrightarrow{g} \mathbb{A}^{p-1} \)

quotient by \( \mathbb{R}^1 \)

\[ [F_L] \in \mathbb{P}(N^c(x)) \]

so \((x)\) tells us that \( F_L \) maps to a point with coordinates in \( \frac{1}{m} \geq \) under \( g \)

\[ \Rightarrow \text{Can't accumulate inside } \mathbb{A}^{p-1} \]

\[ \Rightarrow F_L \text{ can't accumulate inside of } U \]

\[ V = \mathbb{P}(\overline{N, E}(x))_{K + \varepsilon H \leq 0} \leq \mathbb{P}(U) = \mathbb{A}^{p-1} \]

for \( H \) ample compact

so finitely many \([F]\) lie inside \( V \)

\[ \Rightarrow \text{finitely many of the } F_L \text{ lie inside } (K + \varepsilon H \leq 0) \]
\[ V = \text{span}(F) = \mathbb{R} \]

**Step 6**

- **Claim:** If \( F = F \) is closed, then \( F = F \) is a rational subspace.

For any \( k \)-negative \( \alpha \in \mathcal{D}(F) \), we have

\[ \dim F = 1 \]

Use a finite sum argument to show that

\[ \sum_{k=0}^{\infty} F \]

is closed.

So

\[ \overline{\text{NE}(X)} \]

is equal to

\[ \overline{\text{NE}(X)} + \sum_{k=0}^{\infty} F(x) \]

for \( k \to \infty \).

Finite sum closed.
Pick $k$ small enough $k > 0 \ s.t.$

\[ F \setminus \mathfrak{E} \subseteq (K_x + 3H) > 0 \]

\[ W = \overline{NE}_{K+3H > 0} + \sum F_L \]

\[ F_L \neq F \]

Pick $g \in V \ s.t. \ g|_W > 0$

open condition in $V$

since $V$ is rational

\[ \Rightarrow \exists \ g' \ \text{with rational coordinates} \]

\[ s.t. \ g'|_F = 0, \ g'|_W > 0 \]

\[ g' = \left[ \frac{1}{m} D \right] \] for $D$ an integral divisor

so $D|_F = 0, \ D|_W > 0$

\[ \Rightarrow D \oplus \mathfrak{E}, \ D \perp \overline{NE}(X) = F \]
Step 7

\[ mD - (K_X + D) + D \text{ nef} \]
\[ > 0 \quad \underline{\text{on } F} \]
\[ = 0 \quad \underline{\text{on } F \setminus \text{smooth}} \]
\[ > 0 \quad \underline{\text{else}} \]

so for large \( m \), \( mD - (K_X + D) \) is ample.

So by base point free theorem,

For \( b > 0 \), \( \Psi_b : X \to Z \) (Itaka fibrations)

Stein factorize and take \( b \) large.

Call this Itaka fibration

\( \Psi_F : X \to \mathbb{P}^1 \) projective with \( \Psi_F^* O_{\mathbb{P}^1} = O_{X} \)

\( \Psi_F(c) = 0 \iff C \cdot D = 0 \) by semiample

\( \iff [C] \in F \)

so \( \Psi_F \) is as in part c) of the theorem, but this uniquely determines \( \Psi_F \).