Def \( Y \) has terminal singularities if \( K_Y \) is \( \mathbb{Q} \)-Cartier and

for any resolution \( f: X \to Y \)

\[
K_X \cong \mathbb{Q}^e K_Y + \sum a_i E_i, \quad a_i > 0
\]

\( E = E_{\text{exc}}(f) = \bigcup E_i \), \quad \alpha_i = \alpha(E_i, Y)

Example: if \( Y \) is smooth, then \( Y \) has terminal singularities

\( Y \) is independent of choice of resolution

\[
k(Y) = \text{function field of } Y
\]

\( E \cong X \xrightarrow{f} Y \) \( f \) birational \( E \cong X \) divisor

\( \text{ord}_{E} f = \nu(E, x): k(X) = k(Y) \) is

\( \nu(E, x) \Rightarrow \alpha(E, Y) \)

\( \text{a discrete valuation} \)

but if \( E \xrightarrow{g} Y \) \( g \) isomorphism over the generic point of \( E, E' \)

\[
\nu(E', x') = \nu(E, x)
\]

\( \Rightarrow \alpha(E, Y) = \alpha(E', Y) \)
$E$ is a divisor lying over $Y$ up to equivalence coming from inducing the same valuation $N(E_{x})$.

$\text{center } (E) = \text{ closure of } f(\mathcal{E}) \subseteq Y$

**Lemma** $Y$ has terminal singularities if and only if there is some $f: X \to Y$ resolution such that

$$f^{*} \mathcal{O}_{X}(mk_{Y} - E) = \mathcal{O}_{X}(mk_{Y})$$

$mk_{Y}$ is Cartier and $E = \sum E_{i}$ reduced exceptional.

**Proof**

$mK_{X} = f^{*}mk_{Y} + \sum ma_{i}E_{i}$

$\exists ma_{i} \geq 0$ $ma_{i} > 1$

$mK_{X} - E = f^{*}mk_{Y} + \sum (ma_{i} - 1)E_{i}$

$ma_{i} - 1 \geq 0$, then proceed as before.

**Summary**

smooth $\leq$ terminal $\leq$ canonical

determined by $a(E_{j}, Y)$.
Ex 1) Terminal surfaces are smooth.

$\mathcal{F}: X \to Y$ resolution, but $Y$ is terminal, dim $Y = 2$

$E = \bigcup E_i \quad (E_i, E_j)$ negative definite

$K_X \sim Q f^* K_Y + \sum a_i E_i \quad a_i > 0$

Negative definite $\implies$ there exists $E_j$

$s.t. \quad \left(\sum a_i E_i\right) \cdot E_j < 0$

$K_X \cdot E_j = (f^* K_Y + \sum a_i E_i) \cdot E_j < 0$

$E_j^2 < 0 \implies E_j \cong \mathbb{P}^1$ is a (-1) - curve

so our resolution factors as

$X \xrightarrow{g} X^1 \xrightarrow{\phi} Y$

blow up a point

$a_i \in \mathbb{Z}$

$E_j = \text{Exc}(\phi), \quad X^1$ is smooth

by Castelnuovo
2) $C \subseteq \mathbb{P}^2$ degree $n$ curve

$X = \text{cone}(C \subseteq \mathbb{P}^2) \subseteq \mathbb{A}^3$

$E^2 = \text{deg } N_{\mathbb{P}(\mathcal{O}(1))}/C = \text{deg}_C \mathcal{O}(1) = -n$

$(K_x + \mathcal{E})|_E = K_E = 2g - 2 = n(n - 3)$

$K_x = f^* K_x + a \mathcal{E}$

$K_x \cdot E = n(n - 3) + n = n(n - 2)$

$(f^* K_x + a \mathcal{E}) \cdot E = -an$

$a = 2 - n$

\[
\begin{array}{c|c}
 n = 1 & \text{smooth cone } a = 1 \\
 n = 2 & \text{quadric cone } y^2 - xz \quad a = 0 \\
 n = 3 & \text{log canonical singularity} \quad a = -1 \\
 n > 4 & \text{bad}
\end{array}
\]
3) $S \subseteq \mathbb{P}^3$ of degree $n$

$\text{To solve}= \bar{x} \Rightarrow x = \text{cone}$

$E \cong S$

$log \text{ discrepancy}$

$$K_x + E \cong \mathcal{F}^* S \cdot K_x + (1+a)E$$

$$\left(\frac{K_x + E}{E}\right)_{|E} = K_S = \mathcal{O}_S (n-4) = (K_{\mathbb{P}^3}+S)_{|S}$$

$$\left(\mathcal{F}^* K_x + (1+a)E\right)_{|S} = (1+a)E_{|E} = \mathcal{O}_S (-1-a)$$

$$1+a = 4-n \quad \Rightarrow \quad a = 3-n$$

$n=2$ \hspace{1cm} Cone over quadric surface

$$3x^4 - 2w = 0^3 \subseteq \mathbb{A}^4$$

but $a = 3-2 = 1$

singular but terminal

$n=3$

$a = 0$ \hspace{1cm} Cone (cubic)

klt sing

$n=4$

log canonical sing $a = -1$

$n>5$

bad (klt singularities are rational)
log pairs

\[(x, D) \quad D = \sum a_i D_i \quad a_i \in \mathbb{Q}\]

\(Q\) - Weil divisor

Def 1) \((x, D)\) is a log pair if

1) \(k_x + D\) is \(Q\)-Cartier

2) \(D\) is a boundary if

\[0 \leq a_i \leq 1\]

round up \(\lceil D \rceil = \sum \lceil a_i \rceil D_i\) \(\in\) Weil divisors

round down \(\lfloor D \rfloor = \sum \lfloor a_i \rfloor D_i\)

\(D\) is reduced if \(D = \lfloor D \rfloor\)

and \(D\) is boundary

\(k_x + D\) will be the main focus generalizing \(k_x\) if \(D = 0\)

log canonical divisor

if \(x\) is smooth

\(D\) is smooth

\(\Sigma(\log D) = \frac{dx_1}{x_1} \wedge \ldots \wedge dx_n\)

\(\Omega(X, k_x + D) = \Omega_X(0) = \wedge \Sigma(\log D)\)
\[ D = \xi^2 \]

**Philosophy**

- \( U \) smooth but not proper
- \( U \subseteq X \) compactification with \( D = X \setminus U \) simple normal crossings, then \( \Omega^1_X(\log D) \) are invariants of \( U \)

**Motivation for us**

1) **Flexibility:** e.g. \( K_{x+D} \) is \( \mathbb{Q} \)-Cartier even if \( K_X \) not,

2) if \( K_x = 0 \) but we can add some boundary \( D \subseteq X \) and now log \( \text{map} \) for \( K_{x+D} \) can help understand \( X \)

3) **Adjunction** \( (K_x + D)|_D = K_D \)

4) canonical bundle formula for \( K \) trivial fibrations
\[(\log) \text{ discrepancies} \quad (x,D) \text{ log pair} \]

\[\varphi : Y \to X \quad \text{log resolution} \quad (\varphi^{-1}D \cup \text{Exc}(\varphi)) \quad \text{simple normal crossings as} \]

\[E = \bigcup E_i \]

\[Y \setminus E \cong X \setminus \varphi(E) \]

\[\varphi^*\mathcal{O}_X(m(K_X + D)) \mid_{Y \setminus E} = \mathcal{O}_Y(m(K_Y + \varphi^{-1}_*D)) \mid_{Y \setminus E} \]

\[\exists \text{ unique} \quad a(E_i, x, D) \in \mathbb{Q} \]

\[\text{such that} \quad m(K_Y + \varphi^{-1}_*D) \sim \varphi^*(m(K_X + D)) + \sum_{E_i \text{ exc}} a(E_i; x, D) E_i \]

\[K_Y + \varphi^{-1}_*D \sim Q \quad \varphi^*(K_X + D) + \sum_{E_i \text{ exc}} a(E_i; x, D) E_i \]

\[a(E_i, x, D) \text{ is the discrepancy at } p \text{ prime divisor of } x \text{ not exceptional} \]

\[a(p; x, D) = -a_i \text{ if } p = D_i \]
or 0 else \( \gamma(X,D) \)

\[ K_Y \sim Q F^*(K_X + D) + \sum_{p \text{ prime}} a(p, x, D) P \]

\[ b(E_i, x, D) = 1 + a(E_i, x, D) \]

Using numerical equivalence

\[ K_Y \equiv F^*(K_X + D) + A \]

\( \gamma(X,D) \) is unique by being the unique \( A \) s.t.

\[ F^* A = -D \] (negativity lemma)

Exercises:

1. \( D' \) effective \( \mathbb{Q} \)-Cartier divisor, for any prime \( p \)
   
   lying over \( x \), \( a(p, x, D) > a(p, x, D') \)

   strict iff \( \text{center}_x(P) \leq D' \)
2) $X$ smooth, $\mathbb{Z} \leq X$ irreducible
\[ H^2(X, \mathbb{Z}) \rightarrow H^2(X) \]
\[ E \rightarrow \mathbb{Z} \]
\[ a(E, x, D) = k - 1 - \sum a_i \text{ mult}_2 D_i \]
\[ k = \text{codim} (\mathbb{Z} \leq X) \]

3) $\mathcal{F} : Y \rightarrow X$ proper birational
\[ D_Y, D_X \text{ s.t.} \]
\[ \mathcal{F}_* D_Y = D_X \quad \mathcal{O}_Y + D_Y = \mathcal{F}_* (K_X + D_X) \]

then for any prime $\mathcal{P}$ lying over $X$, $Y$,
\[ a(\mathcal{P}, Y, D_Y) = a(\mathcal{P}, X, D_X) \]