1 Hom schemes

Let $X$ and $Y$ be two schemes over $S$. The hom functor $\text{Hom}_S(X, Y) : \text{Sch}_S \to \text{Set}$ is given by

$$T \mapsto \{\text{morphisms } X_T \to Y_T \text{ over } T\}.$$ 

**Theorem 1.** Suppose $X$ and $Y$ are projective over $S$ with $X \to S$ flat. Then $\text{Hom}_S(X, Y)$ is representable by a quasi-projective scheme $\text{Hom}_S(X, Y)$ over $S$.

**Proof.** Given $f : X_T \to Y_T$, we have the graph $\Gamma_f : X_T \to X_T \times_T Y_T = (X \times_S Y)_T$ which is a closed embedding. Now $\text{im}(\Gamma_f) \cong X_T$ is flat over $T$ by assumption so it defines a map $T \to \text{Hilb}_{(X \times_S Y)/S}$. This construction is compatible with basechange so we obtain a natural transformation of functors

$$\text{Hom}_S(X, Y) \to \text{H}_{(X \times_S Y)/S}.$$

Since a morphism is determined by its graph, this is a subfunctor. Moreover, we can characterize the graphs of morphisms as exactly those closed subschemes $Z \subset X_T \times_T Y_T$ such that the projection $Z \to X_T$ is an isomorphism. This identifies $\text{Hom}_S(X, Y)$ with the subfunctor of $\text{H}_{(X \times_S Y)/S}$ given by

$$T \mapsto \{\text{closed subsets } Z \subset X_T \times_T Y_T \mid Z \to T \text{ flat and proper, } Z \to X_T \text{ is an isomorphism}\}.$$ 

We will prove this is representable by an open subscheme of $\text{Hilb}_{(X \times_S Y)/S}$.

We can consider the universal family $Z \to \text{Hilb}_{(X \times_S Y)/S}$ which is a closed subscheme of $X \times_S Y \times_S \text{Hilb}_{(X \times_S Y)/S}$. Then $Z$ comes with a projection $\pi : Z \to X \times_S \text{Hilb}_{(X \times_S Y)/S}$. Now we consider the diagram

$$\begin{array}{ccc}
Z & \xrightarrow{\pi} & X \times_S \text{Hilb}_{(X \times_S Y)/S} \\
\downarrow q & & \downarrow p \\
\text{Hilb}_{(X \times_S Y)/S} & \xrightarrow{p} & \text{Hilb}_{(X \times_S Y)/S}.
\end{array}$$

Then the required open subscheme is given by the following lemma.

**Proposition 1.** Let $T = \text{Spec } R$ be the spectrum of a Noetherian local ring and let $0 \in T$ be the closed point. Let $f : X \to T$ be flat and proper and $g : Y \to T$ proper. Let $p : X \to Y$ be a morphism such that $p_0 : X_0 \to Y_0$ is an isomorphism. Then $p : X \to Y$ is an isomorphism.
Proof. Since $X$ is proper and $Y$ is separated over $T$, the morphism $p : X \to Y$ must be proper. Moreover, since $g$ is proper, every closed point of $Y$ lies in $Y_0$. Furthermore, since $p_0$ is an isomorphism, then $p$ has finite fibers over closed points of $Y$ so $p$ is quasi-finite. Indeed since $p$ is proper, the fiber dimension is upper-semicontinuous on $Y$ and it is 0 on closed points. Therefore $r$ is finite, and in particular, affine. This implies that $R^i p_* \mathcal{F} = 0$ for any coherent sheaf $\mathcal{F}$ and $i \geq 1$. Now the result follows if we know that $f$ is flat. Indeed in this case, $p_* \mathcal{O}_X$ is locally free of rank one by cohomology and base change. On the other hand, the natural map $\mathcal{O}_Y \to p_* \mathcal{O}_X$ is an isomorphism at all closed points $y \in Y_0 \subset Y$ and since both source and target are line bundles, it must be an isomorphism. Then, since $p$ is affine, we have

$$X = \text{Spec}_Y f_* \mathcal{O}_X = \text{Spec}_Y \mathcal{O}_Y = Y.$$  

Thus, it suffices to prove the following that $p$ is flat. We will use the following lemma.

**Lemma 1.** Let $p : X \to Y$ be a morphism of locally Noetherian $T$-schemes over a locally Noetherian scheme $T$. Let $x \in X$ a point in the fiber $X_t$ for $t \in T$ and set $y = p(x)$ its image in the fiber $Y_t$. Then the following are equivalent:

1. $X$ is flat over $T$ at $x$ and $p_t : X_t \to Y_t$ is flat at $x \in X_t$;
2. $Y$ is flat over $T$ at $y$ and $p$ is flat at $x \in X$.

Proof. Consider the sequence local ring homomorphisms

$$\mathcal{O}_{T,t} \to \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}.$$  

Let $I = m_t \mathcal{O}_{Y,y}$. Suppose (1) holds. Then $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{T,t}$ module and $\mathcal{O}_{X,x} / I \mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{Y,y} / I$-module. Consider the composition

$$m_t \otimes \mathcal{O}_{X,x} \to I \otimes \mathcal{O}_{X,x} \to \mathcal{O}_{X,x}.$$  

The first map is surjective by right exactness of tensor products and the composition is injective by since $\mathcal{O}_{X,x}$ is flat over $\mathcal{O}_{T,t}$ so both maps are in fact injections. Thus

$$\text{Tor}^1_{\mathcal{O}_{Y,y}}(\mathcal{O}_{Y,y} / I, \mathcal{O}_{X,x}) = 0. \quad (1)$$  

Since $I \subset m_y$, then $I$ annihilates the residue field $k(y)$ and one can check that Equation (1) and the assumptions imply that $\text{Tor}^1_{\mathcal{O}_{Y,y}}(k(y), \mathcal{O}_{X,x}) = 0$ by the following lemma we will leave as an exercise.

**Lemma 2.** Suppose $R$ is a Noetherian ring and $I \subset R$ is a proper ideal. Let $M$ be an $R$-module such that $M / IM$ is a flat $R / I$-module and such that

$$\text{Tor}^1_R(R / I, M) = 0.$$  

Then for any $I$-torsion $R$-module $N$,

$$\text{Tor}^1_R(N, M) = 0.$$  

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Then \(\mathcal{O}_{X,x}\) is a flat \(\mathcal{O}_{Y,y}\)-module by the local criterion for flatness.

Since everything is local, \(\mathcal{O}_{X,x}\) is in fact faithfully flat over \(\mathcal{O}_{Y,y}\). Now we want to show that \(\text{Tor}_{1}^{\mathcal{O}_{T,t}}(k(t), \mathcal{O}_{Y,y}) = 0\). Pulling back the sequence

\[0 \rightarrow m_t \rightarrow \mathcal{O}_{T,t} \rightarrow k(t) \rightarrow 0.\]

to \(\mathcal{O}_{Y,y}\) gives us

\[0 \rightarrow \text{Tor}_{1}^{\mathcal{O}_{T,t}}(k(t), \mathcal{O}_{Y,y}) \rightarrow m_t \otimes \mathcal{O}_{Y,y} \rightarrow I \rightarrow 0.\]

Since \(\mathcal{O}_{X,x}\) is flat over \(\mathcal{O}_{Y,y}\), then

\[0 \rightarrow \text{Tor}_{1}^{\mathcal{O}_{T,t}}(k(t), \mathcal{O}_{Y,y}) \otimes \mathcal{O}_{X,x} \rightarrow m_t \otimes \mathcal{O}_{X,x} \rightarrow I \otimes \mathcal{O}_{X,x} \rightarrow 0.\]

We saw above that the second map is injective so

\[\text{Tor}_{1}^{\mathcal{O}_{T,t}}(k(t), \mathcal{O}_{Y,y}) \otimes \mathcal{O}_{X,x} = 0\]

but \(\mathcal{O}_{X,x}\) is faithfully flat over \(\mathcal{O}_{Y,y}\) so \(\text{Tor}_{1}^{\mathcal{O}_{T,t}}(k(t), \mathcal{O}_{Y,y}) = 0\) and \(\mathcal{O}_{Y,y}\) is flat over \(\mathcal{O}_{T,t}\).

For the converse, suppose (2) holds. Then \(Y \rightarrow T\) is flat at \(y \in Y\) and \(p : X \rightarrow Y\) is flat at \(x \in X\) so the composition \(X \rightarrow T\) is flat at \(x \in X\). Moreover, \(p_t\) is the pullback \(p\) to \(Y_t\) and flatness is stable under basechange so \(p_t\) is flat at \(x \in X\).

\[\square\]

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**Corollary 1.** Let \(f : X \rightarrow T\) be flat and proper and \(g : Y \rightarrow T\) proper over a Noetherian scheme \(T\). Let \(p : X \rightarrow Y\) be a morphism. Then there exists an open subscheme \(U \subset T\) such that for any \(T'\) and \(\varphi : T' \rightarrow T\), \(\varphi\) factors through \(U\) if and only if \(\varphi^*p : X_{T'} \rightarrow Y_{T'}\) is an isomorphism.

**Proof.** The locus where \(p : X \rightarrow Y\) is an isomorphism is open on the target \(Y\) so let \(Z \subset Y\) be the closed subset over which \(p\) is not an isomorphism. Since \(g\) is proper, \(g(Z) \subset T\) is closed. Let \(U \subset T\) be its complement. By the proposition, a point \(t \in T\) is contained in \(U\) if and only if the map on the fibers \(p_t : X_t \rightarrow Y_t\) is an isomorphism. Since this is a fiberwise condition on \(t \in T\), it is clear that \(U\) satisfies the required universal property.

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## 2 Castelnuovo-Mumford regularity

We will now discuss the first main ingredient in the proof of representability of Hilbert and Quot functors.

**Theorem 2.** (Uniform Castelnuovo-Mumford regularity) For any polynomial \(P\) and integers \(m, n\), there exists an integer \(N = N(P, m, n)\) such that for any field \(k\) and any coherent subsheaf of \(\mathcal{F} \subset \mathcal{O}_{\mathbb{P}^m_k}\) with Hilbert polynomial \(P\) we have the following. For any \(d \geq N\),

1. \(H^i(\mathbb{P}^m_k, \mathcal{F}(d)) = 0\) for all \(i \geq 1\),
2. \( F(d) \) is generated by global sections, and

3. \( H^0(\mathbb{P}^n_k, F(d)) \otimes H^0(\mathbb{P}^n_k, \mathcal{O}(1)) \rightarrow H^0(\mathbb{P}^n_k, F(d+1)) \) is surjective.

To prove this we will define a more general notion of the Castelnuovo-Mumford regularity of a sheaf \( F \) on projective space.

**Definition 1.** (CM regularity) A coherent sheaf \( F \) on \( \mathbb{P}^n_k \) is said to be \( m \)-regular if

\[
H^i(\mathbb{P}^n_k, F(m-i)) = 0
\]

for all \( i > 0 \).

The notion of CM regularity is well adapted to running inductive arguments by taking a hyperplane section.

**Proposition 2.** Let \( F \) be \( m \)-regular. Then

1. \( H^i(\mathbb{P}^n_k, F(d)) = 0 \) for all \( d \geq m - i \) and \( i > 0 \), that is, \( F \) is \( m' \)-regular for all \( m' \geq m \),

2. \( H^0(\mathbb{P}^n_k, F(d)) \otimes H^0(\mathbb{P}^n_k, \mathcal{O}(1)) \rightarrow H^0(\mathbb{P}^n_k, F(d+1)) \) is surjective for all \( d \geq m \).

3. \( F(d) \) is globally generated for all \( d \geq m \), and

**Proof.** The definition of \( m \)-regularity and the conclusions of the proposition can all be checked after passing to a field extension since field extensions are faithfully flat so we may suppose the field \( k \) is infinite. Now we will induct on the dimension \( n \).

If \( n = 0 \) the results trivially hold since all higher cohomology vanishes, all sheaves are globally generated and \( \mathcal{O}(1) = \mathcal{O} \). Suppose \( n > 0 \) and let \( H \subset \mathbb{P}^n_k \) be a general hyperplane.\footnote{Here general means that \( H \) avoids all associated points of \( F \). This is where we use the infinite field assumption.}

Now consider the short exact sequence

\[
0 \rightarrow F(m-i-1) \rightarrow F(m-i) \rightarrow F_H(m-i) \rightarrow 0
\]

where \( F_H = F|_H \) is the restriction. Taking the long exact sequence of cohomology yields

\[
\ldots \rightarrow H^i(\mathbb{P}^n_k, F(m-i)) \rightarrow H^i(\mathbb{P}^n_k, F_H(m-i)) \rightarrow H^{i+1}(\mathbb{P}^n_k, F(m-i-1)) \rightarrow \ldots.
\]

The first and last terms are 0 for all \( i > 0 \) by assumption so \( H^i(H, F_H(m-i)) = 0 \) for all \( i > 0 \). That is, \( F_H \) is \( m \)-regular.

We will continue next time.