1 More on flat morphisms

Last time we left off with the following statement.

**Proposition 1.** Let \( f : X \to Y \) be a morphism of schemes with \( Y \) an integral regular scheme of dimension 1. Then \( f \) is flat if and only if it maps all associated points of \( X \) to the generic point of \( Y \).

**Proof.** Suppose \( f \) is flat and take \( x \in Y \) with \( f(x) = y \) a closed point. Then \( \mathcal{O}_{Y,y} \) is a DVR with uniformizing parameter \( t_y \in m_y \). Since \( t_y \) is a non-zero divisor, \( f^*t_y \in m_x \) is a non-zero divisor so \( x \) is not associated.

Conversely, if \( f \) is not flat, there is some \( x \in X \) with \( y = f(x) \) a closed point and \( \mathcal{O}_{X,x} \) is not a flat \( \mathcal{O}_{Y,y} \) module. Since \( \mathcal{O}_{Y,y} \) is a DVR, this means \( \mathcal{O}_{X,x} \) is not torsion free so \( f^*t_y \) is a zero divisor which must be contained in some associated prime mapping to \( y \).

**Corollary 1.** Let \( X \to Y \) as above. Then \( f \) is flat if and only if for each \( y \in Y \), the scheme theoretic closure of \( X \setminus X_y \) inside \( X \) is equal to \( X \).

The slogan to take away from the above corollary is that flat morphisms over a smooth curve are continuous in the following sense:

\[
\lim_{y \to y_0} X_y = X_{y_0}
\]

for each point \( y_0 \in C \).

**Corollary 2.** Let \( Y \) be as above and \( y \in Y \). Suppose \( X \subset \mathbb{P}_Y^n \) is flat. Then there exists a unique subscheme \( \overline{X} \subset \mathbb{P}_Y^n \) such that \( \overline{X} \to Y \) is flat.

In particular, the functor of flat subschemes of a projective scheme satisfies the valuative criterion of properness!

**Example 1.** Consider the subscheme \( X \subset \mathbb{P}^3_{\mathbb{A}^1 \setminus \{0\}} \) defined by the ideal

\[
I = (a^2(xw + w^2) - z^2, ax(x + w) - yzw, xz - ayw).
\]

For each \( a \neq 0 \), this is the ideal of the twisted cubic which is the image of the morphism

\[
\mathbb{P}^1 \to \mathbb{P}^3
\]

\[
[s, t] \mapsto [t^2s - s^3, t^3 - ts^2, ats^2, s^3].
\]
By the above Corollary, we can compute the flat limit
\[ \lim_{a \to 0} X_a = \overline{X}_0 \]
by computing the closure \( \overline{X} \) of \( X \) in \( \mathbb{P}^3_{\mathbb{A}^1} \). We can do this by taking \( a \to 0 \) in the ideal \( I \) but we have to be careful! Note that the polynomial
\[ y^2w - x^2(x + w) \]
is contained in the ideal \( I \). In fact
\[ I/aI = (z^2, yz, xz, y^2w - x^2(x + w)) \]
which gives the flat limit of this family of twisted cubics. Note that set theoretically this is a nodal cubic curve in the \( z = 0 \) plane but at \( [0,0,0,1] \) it has an embedded point that “sticks out” of the plane.

The following is an interesting characterization of flatness over a reduced base.

**Theorem 1** (somewhere in ega). (Valuative criterion for flatness) Let \( f : X \to S \) be a locally of finite presentation morphism over a reduced Noetherian scheme \( S \). Then \( f \) is flat at \( x \in X \) if and only if for each DVR \( R \) and morphism \( \text{Spec} R \to S \) sending the closed point of \( \text{Spec} R \) to \( f(s) \), the pullback of \( f \) to \( \text{Spec} R \) is flat at all points lying over \( x \).

We will see a proof of this in the projective case soon.

**Proposition 2.** Let \( f : X \to Y \) be a flat morphism of finite type and suppose \( Y \) is locally Noetherian and locally finite-dimensional. Then for each \( x \in X \) an \( y = f(x) \),
\[ \dim_x(X_y) = \dim_x(X) - \dim_y(Y). \]

**Proof.** It suffices to check after base change to \( \text{Spec} \mathcal{O}_{Y,y} \) so suppose \( Y \) is the spectrum of a finite dimensional local ring. We will induct on the dimension of \( Y \). If \( \dim(Y) = 0 \), then \( X_y = X_{\text{red}} \) so there is nothing to check. If \( \dim(Y) > 0 \), then there is some non-zero divisor \( t \in m_y \subset \mathcal{O}_{Y,y} \) so that \( f^*t \in m_x \) is a non-zero divisor. Then the induced map \( X' = \text{Spec} \mathcal{O}_{X,x}/f^*t \to Y' = \text{Spec} \mathcal{O}_{Y,y}/t \) is flat, \( \dim(X') = \dim_x(X) - 1 \), \( \dim(Y') = \dim(Y) - 1 \), and the result follows by induction. \( \square \)

**Corollary 3.** If \( X \) and \( Y \) are integral \( k \)-schemes, then \( n = \dim(X_y) \) is constant for \( y \in \text{im}(f) \) and \( \dim(X) = n + \dim(Y) \).

## 2 Hilbert polynomials

Let \( X \subset \mathbb{P}^n_k \) be a projective variety over a field \( k \). Recall that the Hilbert polynomial of a coherent sheaf \( \mathcal{F} \) on \( X \) may be defined as
\[ P_\mathcal{F}(d) := \chi(X, \mathcal{F}(d)) := \sum_{i=0}^n (-1)^i h^i(X, \mathcal{F}(d)). \]

\(^1\text{It is not a priori clear that this is a polynomial } n. \text{ To prove this, one can induct on the dimension of } X \text{ and use the additivity of Euler characteristics under short exact sequences.}\)
where $\mathcal{F}(d) = \mathcal{F} \otimes \mathcal{O}_X(1)^\otimes d$. By the Serre vanishing theorem,

$$\chi(X, \mathcal{F}(d)) = \dim H^0(X, \mathcal{F}(d))$$

for $n \gg 0$. When $\mathcal{F} = \mathcal{O}_X$, then we call $P_X(d) := P_{\mathcal{O}_X}(d)$ the Hilbert polynomial of $X$.

We have the following important theorem.

**Theorem 2.** Let $f : X \to Y$ be a projective morphism over a locally Noetherian scheme $Y$. If $\mathcal{F}$ is a coherent sheaf on $X$ which is flat over $Y$, then the Hilbert polynomial $P_{\mathcal{F}|_{X_y}}(d)$ is locally constant for $y \in Y$. If $Y$ is reduced, then the converse holds.

**Proof.** By pulling back along the inclusion $\text{Spec} \mathcal{O}_{Y,y} \to Y$, we may assume that $Y = \text{Spec} A$ is the spectrum of a Noetherian local ring. Moreover, by considering the pushforward $i_* \mathcal{F}$ under the map $i : X \hookrightarrow \mathbb{P}^n_Y$, we may assume that $X = \mathbb{P}^n_A$. We have the following lemma:

**Lemma 1.** $\mathcal{F}$ is flat over $Y$ if and only if $H^0(X, \mathcal{F}(d))$ is a finite free $A$-module for $d \gg 0$.

**Proof.** $\Longrightarrow$: Let $\mathcal{U} = \{U_i\}$ be an affine open covering of $X$ and consider the Čech complex

$$0 \to H^0(X, \mathcal{F}(d)) \to C^0(\mathcal{U}, \mathcal{F}(d)) \to C^1(\mathcal{U}, \mathcal{F}(d)) \to \ldots \to C^n(\mathcal{U}, \mathcal{F}(d)) \to 0.$$  

By Serre vanishing, this sequence is exact for $d \gg 0$. Since $\mathcal{F}$ is flat, each term $C^i(\mathcal{U}, \mathcal{F}(d))$ is a flat finitely generated $A$-module. We repeatedly apply the following fact: if $0 \to A \to B \to C \to 0$ is exact and $B$ and $C$ are flat, then $A$ is flat. It follows that $H^0(X, \mathcal{F}(d))$ is a finitely generated flat module over the local ring $A$, and in particular, is free.

$\Longleftarrow$: Suppose that $d_0$ is such that $H^0(X, \mathcal{F}(d))$ is finite and free for $d \geq d_0$ and consider the $S = A[x_0, \ldots, x_n]$ module

$$M = \bigoplus_{d \geq d_0} H^0(X, \mathcal{F}(d)).$$

Now $M$ is $A$-flat since it’s a direct sum of flat modules. Furthermore, $M$ defines a quasicoherent sheaf $\tilde{M}$ on $X$ which is just $\mathcal{F}$ itself. Explicitly, $\tilde{M}$ is obtained by gluing together the degree 0 parts of the localizations of $M$ by each $x_i$. Since flatness is preserved by localization and direct summands of flat modules are flat, we conclude that $\tilde{M} = \mathcal{F}$ is flat. □

Now the first part of the theorem would follow if we know that the rank of the $A$-module $H^0(X, \mathcal{F}(d))$ equals $P_{\mathcal{F}|_{X_Y}}(d)$. This is implied by the following equality base change statement.

$$H^0(X, \mathcal{F}(d)) \otimes_A k(y) = H^0(X_y, \mathcal{F}(d)|_{X_y})$$  \hspace{1cm} (1)

**Remark 1.** One can rewrite equality $\mathcal{F}$ as saying the natural map

$$u^* f_* (\mathcal{F}(d)) \to f'_* u'^* (\mathcal{F}(d))$$

where

$$\begin{array}{ccc}
X' & \xrightarrow{u'} & X \\
\downarrow f' & & \downarrow f \\
Y' & \xrightarrow{u} & Y
\end{array}$$  \hspace{1cm} (2)

is the Cartesian diagram with $i : Y' = \text{Spec} k(y) \to Y$ the inclusion. More generally, given any Cartesian diagram as above and any quasicoherent sheaf $\mathcal{F}$ on $X$, there are natural maps

$$u^* R^i f_* (\mathcal{F}) \to R^i f'_* (u'^* \mathcal{F}).$$
One can ask more generally if this map is an isomorphism, and if it is we say that base change holds (for this diagram, this sheaf, and this $i$), or that the $i$th cohomology of $F$ commutes with base change by $u$. We highlight this here since this situation will come up again.

Suppose first that $y \in Y$ is a closed point. Then consider a resolution of $k(y)$ of the form
\[ A^m \to A \to k(y) \to 0. \]
(3)
Pulling back and tensoring with $F$ we get a resolution
\[ F^m \to F \to F|_{X_y} \to 0. \]
For $d \gg 0$ and by Serre vanishing, the sequence
\[ H^0(X, F(d)^{\oplus m}) \to H^0(X, F(d)) \to H^0(X_y, F(d)|_{X_y}) \to 0 \]
is exact. On the other hand, we can tensor sequence 3 by the $A$-module $H^0(X, F(d))$ to get an exact sequence
\[ H^0(X, F(d)^{\oplus m}) \to H^0(X, F(d)) \to H^0(X, F(d)) \otimes_A k(y) \to 0. \]
Comparing the two yields the required base change isomorphism. Now if $y$ is not a closed point of $Y$, we can consider the Cartesian diagram as in 4 where $Y' = \text{Spec } O_{Y, y}$. Then $u$ is flat and $y$ is a closed point of $Y'$ and we can reduce to this case by applying the following.

**Lemma 2.** (Flat base change) Consider the diagram
\[ \begin{array}{ccc}
X' & \xrightarrow{u'} & X \\
\downarrow{f'} & & \downarrow{f} \\
Y' & \xrightarrow{u} & Y
\end{array} \]
where $f$ is qcqs and $u$ is flat and let $F$ be a quasi-coherent sheaf on $X$. Then the base change morphism
\[ u^* R^i f_*(F) \to R^i f'_*(u'^* F). \]
is an isomorphism for all $i \geq 0$.

Now when $Y$ is a reduced local ring, a module $M$ is free if and only if $\dim M_y$ is independent of $y$ for each $y \in Y$ so using the now proven base change isomorphism
\[ H^0(X, F(d)) \otimes_A k(y) = H^0(X_y, F(d)|_{X_y}) \]
we obtain that $H^0(X, F(d))$ is a finite free $A$-module if and only if $P_{F|_{X_y}}(d)$ is independent of $y \in Y$.

**Remark 2.** As a corollary, we obtain the valuative criterion for flatness in the case of a projective morphism since the constancy of the Hilbert polynomial can be checked after pulling back to a regular curve.

**Remark 3.** The Hilbert polynomial encodes a lot of geometric information about a projective variety $X$ such as the dimension, degree of projective dimension, and arithmetic genus. In particular, these invariants are constant in projective flat families.

\[ ^2 \text{quasi-compact quasi-separated} \]