Mirrors of curves and their Fukaya categories

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Plan: 1. Mirror construction
      2. The fiber-wise wrapped Fukaya category
      3. Homological mirror symmetry for hypersurfaces
      4. Floer theory in trivalent configurations

\{ Abouzaid-A. Ehrman-Katzarkov \}
$H = f^{-1}(0) \subset (\mathbb{C}^*)^n$ or toric variety $V$

hypersurface $(\dim_{\mathbb{C}} n-1)$ (actually, defined over Novikov field $K$ / degenerating family)

$H = f^{-1}(0), \quad f(x) = \sum_{\alpha \in \mathbb{A} \subset \mathbb{Z}^n} c_\alpha \ t^{p(\alpha)} x^\alpha$

Laurent polynomial $(t \to 0)$

$\Rightarrow$ candidate mirror "Landau-Ginzburg model" $(Y, W)$

$Y = \text{toric Calabi-Yau } (n+1)-\text{fold}, \quad W : Y \to \mathbb{C}$ "Super-potential"

(non compact)

expect Fukaya category $\mathcal{F}(H) \leftarrow \sim \rightarrow \mathcal{D}_{\text{sg}}^b (Y, W)$

and coherent sheaves $\mathcal{D}_{\text{coh}}^b (H) \leftarrow \sim \rightarrow \mathcal{F}(Y, W)$

Abouzaid - A. (see also Cannizzo)

for abelian var.
Construction: (Abouzaid - A.- Katzarkov, see earlier work of Hori-Vafa, P. Clarke, ... also Chan-Lau-Leung, ...)

\[ H = f^{-1}(0) \subset (\mathbb{R}^n)^n, \quad f(x) = \sum_{\alpha \in A \subset \mathbb{Z}^n} c_{x} \exp^{\langle \alpha, x \rangle} \times \alpha \text{ Laurent polynomial } (t \to 0) \]

(or \( H = f^{-1}(0) \subset V \) tonic var., \( f \) section of \( \mathbb{Z} \to V \) with associated polytope \( \text{Conv Hull}(A) \)).

\[ \implies \text{to find mirror } (Y, W), \quad \varphi = \text{Trop}(F) : \mathbb{R}^n \to \mathbb{R}, \quad \varphi(\xi) = \max_{\alpha \in A} \langle \alpha, \xi \rangle - c(\alpha). \]

Let \( Y = \) tonic Kähler (CY) var. with moment polytope \( \Delta_Y = \{(\xi, \eta) \in \mathbb{R}^{n+1}/\eta \geq \varphi(\xi)\} \) and \( W = w_0 = -z^{(0, 0, 0, 1)} \) tonic monomial vanishing to order 1 on each tonic divisor \( \subset Y \).

(\( \text{rep. } W = w_0 + W_V : W_V = \text{one monomial for each tonic divisor of } V \))

\[ \text{Ex. 1: } \]

\[ H: \{1 + x_1 + x_2 = 0\} \implies \begin{array}{c}
\xi_2 \\
\downarrow \\
0 \\
\xi_1 \\
\end{array} \]

\[ \Delta_Y: \eta \geq \max(0, \xi_1, \xi_2) \]

\[ \eta = \mathbb{C}^3, \quad w_0 = -z_1z_2z_3 \]

\[ \text{Ex. 1': } \]

\[ \{x_0 + x_1 + x_2 = 0\} \subset \mathbb{P}^2 \implies (\mathbb{C}^3, \frac{-z_1z_2z_3 + T(z_1 + z_2 + z_3)}{w_0}, \frac{w_V}{w_0}) \]
Remark: \( \text{for } c \neq 0, \, w_0^{-1}(c) = (\mathbb{C}^*)^n \) mirror to ambient \( (\mathbb{C}^*)^n \),

equipped with \( u_v \rightsquigarrow \text{mirror to toric var } V. \)

- \( w_0^{-1}(0) = U \text{ toric var's } \) (one for each term in \( f \)),

interseeding along \( \text{crit}(w_0) = U \text{ (n-1)-dim. strata}. \)

- the monodromy of \( w_0 \) around origin is "mirror to \(-\Theta \Theta(\mathcal{H})\)."

\[
\begin{align*}
\text{Ex. 2:} & \quad H: \begin{cases} 
1 + x_1 + x_2 + \frac{t}{x_1 x_2} = 0^3 
\end{cases} & \rightarrow & \begin{cases} 
\xi_2 
\end{cases} \\
\text{Ex. 2':} & \quad V & \rightarrow & \begin{cases} 
\xi_1, \xi_2 
\end{cases} \\
\end{align*}
\]

\( y = \text{Tot} \left( O(-3) \rightarrow \mathbb{C} \mathbb{P}^2 \right) \)

\( \text{u, } (z_0, z_1, z_2) \)

\( w_0 = -u z_0 z_1 z_2 \)

\( (\mathbb{Y}, w_0 + u_v = -u z_0 z_1 z_2 + T u (z_0^3 + z_1^3 + z_2^3)) \)

Morse-Bott along smooth cubic elliptic curve

\( \cong \text{usual mirror. } \)
Ex. 3: \( H = \text{genus } g \geq 2 \text{ curve } \subset \text{toric surface } \rightarrow (y, w_0 + w_v) \)

\[ \text{crit}(w_0 + w_v) = \text{trivalent configuration of } (3g-3) \mathbb{CP}^1 \text{'s meeting in } (2g-2) \text{ nodes} \]

E.g. genus 2

\[ \text{crit}(w_0 + w_v) = \]

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Jeff Koons, "Balloon Dog" - photo NY Times
The fiberwise wrapped Fukaya category of \((Y, w_0 + w_V)\) (Abouzaid: A.)

(\(\sim\) in proper case, can use Seidel’s work; we want to allow fiberwise noncompact Lagrangians)

Remark: A Landau-Ginzburg model \((Y, w) \xrightarrow{W} C\) should determine a (non-exact) sector (manifold with boundary), \(Y - \{\text{Re}\, W < 0\}\), and its wrapped Fukaya category (stopped at \(\text{Re}\, W < 0\))

Instead of stops/sectors, we use \textbf{monomial admissibility} (cf. A. Hanlon’s thesis)

**Objects**: properly embedded Lagrangians \(L \subset Y\) (+ extra data: spin str., grading, ...)

which are tautologically unobstructed (bound no hol. discs) + monomially admissible:

1) for \(|w_0| \gg 1\), \(\arg(w_0|_L)\) is loc. constant \(\in (-\pi/2, \pi/2)\) (i.e. \(w_0|_L \in \text{union of radial arcs}\))

2) recalling that fibers of \(w_0\) are \(\approx (\mathbb{C}^\times)^m\), outside of a compact subset of these, there exists an open cover \(\{U_v\}\) and a collection of monomials \(z^v\) [the monomials appearing in \(w_v\) if \(V\) is compact, else pick a toric compactification!]

\(\text{s.t. } \arg(z^v)|_{\partial U_v}\) is loc. constant (e.g. \(\equiv 0\)).

**Note**: *monomial admissibility gives control over discs in Floer products via maximum principle*

*use a specific toric Kahler form for which \(\{\arg w_0, \arg z^v\}\) Poisson-commute in \(U_v\).
Perturbations for $F(Y, w_0 + w_0)$:

$L$ admissible $\implies$ flow $L^t$ (Ham. isotopic to $L$; admissible)

The flow increases the values of $\text{arg}(w_0)$ and $\text{arg}(z^v)_{|U_0}$ at $\infty$.

- for $w_0$ and for $z^v$ which appear in $w_0$, $\text{arg}^+$ in bounded interval $c \in (-\pi, \pi)$

  \[
  \left( \text{the stop at } \infty \text{ is } \{\text{arg}(w_0) = \pi\} \cup \bigcup \{\arg z^v = \pi\} \cap U_0 \right)
  \]

- for other monomials $z^v$ not in $w_0 + w_0$, $\text{arg}^+$ to $\infty$ (WRAP)

Define

\[
\text{hom}(L_0, L_1) := \lim_{t \to \infty} \text{CF}^* (L_0^t, L_1) \quad \text{under natural continuation maps.}
\]

(The fiber-wise wrapping Hamiltonians are smoothings of integer piecewise linear functions of the tonic moment map, as in Hanlon)
Example: \((\mathbb{C}^3, w_0 = -z_1 z_2 z_3)\) (mirror of \(\triangle\))

\[ L_0 = \text{parallel transport} \quad l_0 = (\mathbb{R}^*_+) \subset (\mathbb{C}^*)^2 \simeq w_0^{-1}(-1) \text{ along } U\text{-shaped arc.} \]

\[
\begin{align*}
\psi_0^{-1}(-1) &\subset (\mathbb{C}^*)^2 \\
U &
\end{align*}
\]

\[ l_0 \simeq (\mathbb{R}^*_+)^2 \]

\[
\begin{array}{c}
\text{images of } l_0 \\
\text{under monodromy} \\
\text{+ wrapping Ham.}
\end{array}
\]

\[
\arg(z_i) = f_i(\log|z|)
\]

\[
\begin{array}{c}
\arg(z_i) = 0 \text{ wherever } |z_i| \gg \min_j |z_j| \\
\text{wrapping} \\
t \text{ (here } t \to +\infty) \\
\end{array}
\]

\[ \text{hom}(L_0, L_0) \simeq CW^*(l_0, l_0) \oplus CW^*(l_0, l_0)[-1] \]

\[ \simeq \mathbb{K}[x_1^{\pm 1}, x_2^{\pm 1}] \quad \mathcal{O} = \text{multiplication by } 1 + x_1 + x_2 \]

\[ H^* \text{hom} (L_0, L_0) \simeq \mathbb{K}[x_1^{\pm 1}, x_2^{\pm 1}]/(1 + x_1 + x_2) \simeq \text{hom}(\mathcal{O}, \mathcal{O}) \text{ in } D^b\text{Coh}(\triangle) \checkmark \]

Similar calculation for \(H \subset (\mathbb{C}^*)^n\) hypersurface: \(H^* \text{hom}(L_0, L_0) \simeq \mathbb{K}[x_1^{\pm 1}]/(f) \simeq \text{End}(\mathcal{O}_H) \)

\[ \mathcal{O} = \text{mult. by } f = \text{defining eq. of } H \]

\[ \Rightarrow \text{this gives HMS if } L_0 \text{ generates } F(\gamma, w_0). \]
Example: \((C^3, w_0 + w_V = -z_1 z_2 z_3 + T(z_1 z_2 + z_3))\) (mirror of \(H: x_0 + x_1 + x_2 = 0\)).

Same construction, but don't wrap - only increase \(\arg(z_i)\) slightly.

Fibers \(\{w_0 = -c\}\) are \(\approx ((C^2)^2, w_V = T(z_1 z_2 + \frac{c}{z_1 z_2}))\) \(\approx\) mirror to \(\mathbb{P}^2\).

Start with \(L_k \subset \mathbb{C}^2(-1) \approx (C^2)^2\), \(\arg(z_i) = f_i(\log |z_i|)\) twist by \(2\pi k\) across admissible \(w_V\) mirror to \(\mathcal{O}_{\mathbb{P}^2}(k)\). \(H^* (L_0, L_k) \approx H^*(\mathcal{O}_{\mathbb{P}^2}(k))\).

\(L_k = \) parallel transport \(L_k\) along \(U\)-shaped arc.

Images of \(L_0, L_k\) under monodromy \(\approx L_0, L_{k-1}\)

\(\Rightarrow \hom(L_0, L_k) = \mathcal{C}F^*(L_0, L_k) \oplus \mathcal{C}F^*(L_0, L_{k-1})[-1]\)

\((\text{in general, } \mathcal{C}F = \text{def-sech of } \mathcal{O}(1))\)

\(\Rightarrow H^0 \hom(L_0, L_k) = H^0(\mathbb{P}^2, \mathcal{O}(k))/\langle x_0 + x_1 + x_2 \rangle \approx H^0(\mathcal{O}^H, \mathcal{O}(k)).\) (Abuzzard-At.

(see also Cannizzo.)

In general: match \(U\)-shaped Lagrangians \(\leftrightarrow \mathcal{L}_1 H\) for \(\mathcal{L} \rightarrow \mathcal{V}\) line bundle. \(\Rightarrow\) HMS mod generation.
The Fukaya categories of \((Y, w_0 + w_V)\) and \((F = (\mathbb{C}^n), w_V)\) are related by functors of fiber of \(w_0\):

\[
\mu \: \mathcal{F}(F, w_V) \leftrightarrow \mathcal{F}(Y, w_0 + w_V)
\]

\(\mathcal{U} \subset \mathcal{F}\) (admissible) along \(\mathcal{U}\)-shape

\(\mathcal{N} \subset \mathcal{Y}\) at \(w_0 \to \infty\) (actually \(\in Tw \mathcal{F}(F, w_V)\) if \(w_0\) has more than one end).

Exact triangle of functors on \(\mathcal{F}(F, w_V)\):

\(\mu, S, id\)

(Abozaid-Ganatra, Sylvan)

where \(S = \) section-counting natural transformation from \(\mu\) to \(id\)

\(\mathcal{U} \subset \mathcal{F}\), \(S_0 \subset CF^0(\mu^{-1}(l), l)\) counts holom-sections of \(w_0\) over \(w_0\)

In this language, the above calculation is:

\[
\text{hom}_Y(\mathcal{U} \ell, \mathcal{U} \ell') \simeq \text{hom}_F(l, \mathcal{U} \ell') \simeq \text{Cone}(\text{hom}_F(l, \mu^{-1}(l')) \xrightarrow{S} \text{hom}_F(l, l'))
\]

& homological mirror symmetry is proved by matching this with \(\text{D}^b\text{Coh}(\text{H}) \xrightarrow{\sim} \text{D}^b\text{Coh}(\text{V})\)

\[
\text{hom}_H(\iota^* \mathcal{L}, \iota'^* \mathcal{L}') \simeq \text{hom}_V(\mathcal{L}, \mathcal{L}' \otimes O(-H)) \simeq \text{Cone}(\text{hom}_V(\mathcal{L}, \mathcal{L}' \otimes O(-H)) \xrightarrow{f} \text{hom}_V(\mathcal{L}, \mathcal{L}'))
\]
Theorem (Abouzaid-A.) “in progress”

\[ H \hookrightarrow V \text{ hypersurface in } (\mathbb{K}^*)^n \text{ or toric Fano var.} \]

\[ \Rightarrow \text{ commutative HMS diagram:} \]

\[ \mu \circ \mathcal{F}(F, w_V) \xrightarrow{U} \mathcal{F}(Y, w_0 + w_V) \]

HMS for toric vars. \[ \cong \]

HMS for hypersurfaces (expect \[ \cong \])

\[ \mathcal{O}(H) \cong \text{D}^b\text{Coh}(V) \xrightarrow{i_*} \text{D}^b\text{Coh}(H) \]

Returning to Ex. 2:

\[ Y: \{ 1 + x_1 + x_2 + \frac{t}{x_1 x_2} = 0 \} \]

\[ y = \text{Tot} \left( O(-3) \to \mathbb{CP}^2 \right) \]

\[ w_0 = -u z_0 z_1 z_2 \]

\[ 0 \]

\[ 1 - \xi_1 - \xi_2 \]

\[ V \]

\[ (Y, w_0 + w_V = -u z_0 z_1 z_2 + T u (z_0^3 + z_1^3 + z_2^3)) \]

Morse-Bott along smooth cubic elliptic curve \[ C_T \subset \mathbb{CP}^2 \]

\[ \cong \text{ usual mirror } \]

\[ \Rightarrow \text{ Here } \mathcal{F}(Y, w_0 + w_V) \cong \mathcal{F}(C_T) \text{ via "thimble"} \]

\[ T_0 = S^1 \times \mathbb{R}^2 \subset Y \]

\[ \text{hom}_Y(T_0, T_{l'}) = \text{HF}_C(l, l'). \]

For \[ H = \]

\[ \text{there are similarly mirrors } C = \]

\[ \text{[M. Jeffs].} \]
For Ex. 3: \( H = \text{genus g} \geq 2 \text{ curve } \subset \text{tunec surface } \rightarrow (y, w_0 + w_V) \)

\[
\text{crit}(w_0 + w_V) = \text{trivalent configuration of } (3g - 3) \mathbb{CP}^1's \text{ meeting in } (2g - 2) \text{ nodes}
\]

\[
\text{genus } 2 \quad \bigcirc \quad \bigcirc \quad \rightarrow \text{crit}(w_0 + w_V) = \quad \bigcirc \bigcirc \bigcirc
\]

How does one associate a Fukaya category to a trivalent configuration of Riem. surfaces (\( \mathbb{CP}^1 \& \text{C's} \))? (A-Efimov-Katzarkov)

Naive construction: line bundles \( \leftrightarrow \) thimbles on trivalent graphs?

Looking at mirror of \( \bigcirc \bigcirc \bigcirc \) where \( \text{crit}(w_0) = \bigcup \bigcirc \bigcirc \bigcirc \)

would like to consider \( L_0 = \bigcup \bigcirc \mathbb{R}_{\geq 0} \)

and its "wrapped Floer homology"
$L_0 = \bigcup \{ R \geq 0 \} \subset \bigcup \{ C \}$ and its “wrapped Floer homology”:

- $CW(L_0, \phi(L_0))$ has 3 infinite series of generators in the cylindrical ends + 1 generator at vertex.

$\rightarrow$ should give an additive basis of ring of $\mathbb{F}^\infty$ on $X = \mathbb{P}^1 - \{0, -1, \infty\}$?

- evaluation at points: $L_p = S^1$ in one leg (+ local system)

$\leftrightarrow \Theta_p, p = \text{point on } \bigcirc \text{ near a puncture}$

area enclosed by $S^1 \leftrightarrow \text{valuation of local coord. at puncture}$.

* By analogy with $W(\bigcirc)$ & $W(\bigcirc)$, generators in cyl. ends should be successive powers of inverse of local coord.

**Fact:** 1 and $(\text{local coord.})^{-k} (k \geq 1, i \in \{1, 2, 3\})$ are an additive basis of $\Theta(H)$ (partial fraction!)

$\leftrightarrow \text{generator at vertex} \leftrightarrow \text{in legs}$

* There is no canonical local coord. on $\mathbb{P}^1 - \{0, -1, \infty\}$ near a puncture.

$\overset{\text{HMS}}{\rightarrow} \text{the manner in which (thimble on) } L_p \text{ determines an object of } F(Y, W) \text{ depends on choice of bounding cochain} \ (\text{"framing" in Aganagic-Vafa/Liu/...})$

* Since a function correspond to a generator in one end is nonzero at points near other punctures, Floer trajectories must propagate through the vertex.
A-model data for \( \Sigma = \text{Diagram} \):

- **Preferred choices of** \( \mathbf{P}_i \):
  - for \( \infty \): \( z \) or \( -(z+1) \)
  - for \( 0 \): \( -\frac{z+1}{z} \) or \( \frac{1}{z} \)
  - for \( -1 \): \( -\frac{1}{z+1} \) or \( \frac{-z}{z+1} \)

- Symplectic form \( \omega \) \((\Rightarrow \text{area}(\mathbf{P}_i) = A_i > 0)\)
- \(+B\)-field/bulk deformation \((\Rightarrow \text{weight}(\mathbf{P}_i) \in \text{val}^-((A_j)_{i,K}))\)
- "chart data" at each vertex
  - choice of \( \mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3 \in \text{O}(\mathbb{P}^1 - \{0, -1, \infty\}) \)
  - st. \( \mathbf{p}_i \) extend to meromorphic functions on \( \mathbb{P}^1 \)
  - with a single pole at the respective puncture.

Mirror interpretation: near max. degeneration, \( \text{Diagram} \) = smoothing of

- \( y_i y'_i = T^{A_i} \)
  - smoothing param.
  - or weight of components of \( \Sigma \)

or better,

- \( xy = T^{A/2} \)

- \( \frac{1}{x} = x' y' \)

- \( x' y' = T^{A/2} \)
Objects of $\mathcal{F}(\Sigma)$ (of line bundle type) =
trivalent graphs with one arc on each component of $\Sigma$,
and fixed tangent direction at the vertices
(+ local systems, trivialized at the vertices)

To define $\text{hom}(A,B)$, use $\text{Ham}_e$ perturb (w/min. at vertex) to rotate legs of $A$
slightly ccw at vertices + wrap at $\infty$ in any cylindrical ends of $\Sigma$.
Each vertex gives a degree 0 generator; other intersections as usual.

$\text{Ex: } \text{hom}(L_0, L_0)$:

Interpretation: think of a line bundle $L$ on mirror curve as built by gluing together

$\leftrightarrow \mathcal{O}$ on $\infty$ on $0$ and $\mathcal{O}(k)$ on $\mathbb{P}^1$ with
fixed trivializations at $0$ and $\infty$. 
Differential products on hom(Li,Lj) count rigid (0-dim! families) chains/trees of (Xj-perurbed) holomorphic discs in Σ with boundary on Li & Lj, attached together at vertices of Σ, with weights given by sympl. area (R holonomy of local systems along ∂) and multiplicities:

\[ \text{eg.: } y \xrightarrow{\mu^2} x \]

\[ \text{contributes to coefft of } y \text{ in } \exists x. \]

(+ special rule for \(\mu^2\) output at vertex)

Chart data (loc. coords \(p_i\) for each leg of each vertex) determine the multiplicities:

For a vertex where the incoming strip-like end locally covers \(d_1\)-fold (\(d_1 \geq 1\)) a leg of \(\Sigma\) with local coord. \(p_1\), and the outgoing one locally covers \(d_2\)-fold (\(d_2 \geq 0\)) a leg with local coord. \(p_2\), the local multiplicity contribution := coefficient of \(\frac{1}{p_2-d_2}\) in the power series expansion of \(p_1^{d_1}\) near the pole of \(p_2\). Overall multiplicity = \(\prod\) vertices

Eg. in above example, say \(p_1 = \frac{1}{z+1}\) at \(-1\)  
\[ \frac{1}{z+1} = (1 + \frac{1}{z})^{-1} - (\frac{1}{z})^2 + \ldots \]

\(d_1 = 1, d_2 = 0\)

\(p_2 = \frac{1}{z^2}\) at \(0\)

\[ \Rightarrow \text{multiplicity of } p_2^{-0}. \]

Prop. (A. Efimov-Kutzarkov): \(\| \mu^k \) satisfy the Ao0-relations.
Example: \( \text{hom}(L_0, L_0) \)

\[ \exists x_0 = T^E y - T^f y = 0 \]

\[ \exists x_1 = \pm T^a y \quad \text{Only contribution is} \]

\[ \exists x_2 = \pm T^{\text{something}} y \quad \text{(similar)} \]

\[ \Rightarrow H^4\text{hom} = 0 \quad \text{as expected} \]

\( H^0\text{hom} \) consistent w/ \( \Theta(\infty) \):

\[ \# \text{function with pole order 1}! \]

but \( \exists \) additive basis with one generator for each pole order \( \geq 2 \).

Can in fact get the ring structure on \( H^0\text{hom} \) to match too!

\( \sim \) after change of var's, \( H^0\text{hom}(L_0, L_0) \approx \mathbb{K}[x, y]/y^2 = x^3 + ax + b \)

\[ x \sim x_2 + cx_4 \]

\[ y \sim x_3 + \ldots \]
Theorem (A.-Efimov-Katzarkov, in progress)

The Fukaya category of a trivalent configuration of $C$'s and $\mathbb{P}^1$'s as defined above is derived equivalent to coherent sheaves on the mirror curve constructed by gluing $(\mathbb{P}^1, 3\text{ pts})$ and smoothing as prescribed.

**Example of $\mu^2$ propagating:**

**Special case:** output at a vertex

Expand (partial fraction)

$$P_1^{d_1} P_2^{d_2} = c_0 + \sum_{k>1} a_k P_1^k + \sum_{l>1} b_l P_2^l$$

this is the multiplicity.