Recall: $\Delta \leftrightarrow (\mathbb{C}^3, -\Omega^2)$; we used $U$-shaped Lagrangians but wish to instead work in $\text{wt}(\phi) < 1$; bivalent only of $C$'s $\text{LP}$'s.

Skyscrapers $\phi_1, \phi_2 \in \mathcal{H} \leftrightarrow$ thimbles on $S^1 \subset C$ and axis + boundary exchain (Aganagic-Vafa)

For the strata (Y) $\phi_1$, there's no actual thimble... though there's a precise smooth continuation

$\mathcal{U} \subset \mathbb{R}^\infty \hookrightarrow \mathbb{C}$

\[ \mathcal{U} \subset \mathbb{R}^\infty \hookrightarrow \mathbb{C} \]

parallel transport

6. Floer theory on bivalent configurations of Riem surfaces (e.g. $\phi'$ and $\phi$)

Goal: generalize usual wrapped Floer theory (w/ non-exact Lag's, using Novikov coefficients to record homological energy)

Recall wrapped Floer homology on cylinder:

- $\text{End}(L_0) \approx \mathbb{R}[x^{\pm 1}]$

- $\phi^*(L_0) \approx \text{triangle}$

- evaluation at points $\leftrightarrow$ calc. compositions

The edge on $L_p$ wraps $k$ times around $S^1$ (has "local degree $k$")

On "mirror of pair of pants", $\Sigma = \sqcup \mathbb{C}$

want to consider $L_0 = \mathbb{U}_3 \mathbb{R}_0^{\infty}$

& its "wrapped Floer homology".

- keeping in mind this represents $\text{HW}^0$ of the tropical Lag. pants $\text{HW}^0 L_0 = \mathbb{U}_3 \mathbb{U}_0^{\infty}$

- $\text{CW}(L_0, \phi(L_0))$ has 3 infinite series of generators in the cylindrical ends

This should lead to an additive basis of the ring of functions on $\text{Pl} \setminus \{0, -1, \infty\}

evaluation at points $L_p \ni S^1$ in one leg $\leftrightarrow$ br. system

area enclosed by $S^1$ in $C \leftrightarrow$ valuation of local cusp $C$-function.
(Recall the exact manner in which $L_p$ determines an object in $F(Y, w_0)$ depends on

towering cubic construction - "Bronze" in language of Ag-Vafa-Liu."

This amounts to a choice of local coord. at the end of the pair of pants.

Eg. near $x_1 \to 0$ in pants $\{1 + x_1 + x_2 = 0\} \subset (\mathbb{C})^2$, can use any of $x_1, -\frac{x_1}{x_2}, \frac{x_1}{x_2}, \ldots$

Note $\text{val}(x_1) = \text{val}(\frac{-x_1}{x_2})$

$\text{val} = \text{area enclosed}$

* Leading terms (smallest polynomials, those contained in the cyl-end)

  for products of tentable generators of $CW(\text{loc, loc})$

  evaluation at points

  work just as in cylinder case & suggest the generators in a cylindrical end are

  successive powers of (inverse) of local coordinate (function with $P$ pole order).

  Check: $1, (\text{local coord})^{-k} (k > 1)$ form an additive basis of $O(C)$

  (partial fractions ...).

* However: These functions are nonzero even in the other ends of the pants! So

  there needs to be some holomorphic which cross

  through the node ("propagate").

Propagation rules can be determined by reverse engineering from HRS.

We expect they would calculation in $(Y, w_0)$ (or for tqr logayrjan)

- and they do yield equivalent formulae cut \textit{a posteriori} -- but no direct proof.

---

1. Data: A model data for \( F(Y, w_0) \) consists of

   - symplectic areas of $\mathbb{P}^3$'s: $A_1, A_2, A_3$
   - "chart data" at each node $p_1, p_2, p_3$ leads to coord. on ends of a pair of pants

   $$\text{choice of functions } p_1, p_2, p_3 \in \mathcal{O}(\mathbb{P}^{3} \setminus \{-1, 0, 0\})$$

   s.t. $p_1, p_2, p_3$ extend as meromorphic function on $\mathbb{P}^3$

   with a single pole at the respective points.

   Preferred choices: $p$ takes values $-1$ and $0$ at the other punctures.

   For $\infty$: can take $z$ or $-(z+1)$; For $0$: $\frac{z+1}{z}$ or $\frac{1}{z}$; For $-1$: $\frac{-1}{z+1}$ or $\frac{-z}{z+1}$.
B-side: Interpretation: near max. degeneration, $\infty \rightarrow \infty$ = smoothing of

\[ \text{glue } \mathbb{P}^1 \text{'s at } \{-1,0,\infty\} \text{ wing, for each node, smoothing} \]

\[ \text{of the form } y \psi = \psi(A) \]

\[ y, y' = \text{choice of local coordinates on } \mathbb{P}^1 \text{ near the marked pt} \]

\[ := \text{inverse of the functions chosen above} \]

\[ \psi(A) = \text{area}(\mathbb{P}^1) \text{ smoothing parameter.} \]

- Equivalently but better for us, think of each gluing as:
  \[ \text{glue } \mathbb{P}^1_x \equiv 0 \text{ to parts: } x \psi = \psi(A/2) \]
  \[ \text{& glue } \mathbb{P}^1_x \equiv \infty \text{ to parts, } x \psi = \psi(A/2) \]
  \[ \text{& local coord. } x = \frac{1}{x} \]

The choice of "chart data" reflects the fact of a completely canonical way to glue parts explicitly in this way without choosing local coordinates on the $\mathbb{P}^1$'s near the nodes.

- The chart data will determine "propagation rules" for the Fock differential. The most canonical choice is

\[ \left\{ \begin{array}{ccc} z, & -\frac{z+1}{z}, & -\frac{1}{z+1} \\ \frac{1}{z}, & -\frac{z}{z+1} & \end{array} \right\} \]

or

\[ \left\{ \begin{array}{ccc} -(z+1), & \frac{1}{z}, & -\frac{z}{z+1} \\ \end{array} \right\} \]

which exhibit cyclic symmetry, i.e. can just specify a cyclic ordering of the 3 legs at $\bigstar$. Then

\[ \begin{cases} p_1p_2p_3 = 1 \\ p_1^{-1} + p_{i+1} + 1 = 0 \end{cases} \]

---

2. Objects of $F$ (of line bundle type) = bivalent graphs with
one arc on each component, and fixed direction at the nodes.
(+ local systems on the arcs, trivialized at the node).

To define $\text{hom}(A,B)$, push legs of $A$ at vertices slightly counterclockwise
+ wrap positively in any $A^*$ components.
Then have one degree 0 generator at each vertex + generators at other interactions as usual.

Ex: hom\((L_0, L_0)\) has generators:

Interpretation: think of a line bundle \(\mathcal{L}\) as built by gluing together

\[
\begin{align*}
\text{constant function on } & \quad \mathcal{O} \quad \text{on each} \\
\end{align*}
\]

\[
\begin{align*}
\text{sections of } & \quad \mathcal{O}(k) \quad \text{on } \mathbb{P}^1 \\
& \quad \text{with distinguished trivializations} \\
& \quad \text{at } 0 \text{ and } \infty.
\end{align*}
\]

Generators of \(\text{hom}(L_1, L_2)\) are in correspondence with

\[
\begin{align*}
\text{constant function on } & \quad \mathcal{O}(k) \quad \text{on each} \\
\end{align*}
\]

\[
\begin{align*}
\text{sections of } & \quad \mathcal{O}(k) \quad \text{on } \mathbb{P}^1 \\
& \quad \text{or rather, } H^*(\mathbb{P}^1, \mathcal{L}_1^* \otimes \mathcal{L}_2 \otimes \mathcal{O}(-p_0 - p_\infty))
\end{align*}
\]

* when \(k > 0\), the generators at end points correspond to the monomials which don't vanish at \(0\) and \(\infty\), i.e. \(x^k\) and \(x^{-k}\).

* when \(k \leq 0\), we'll have Floer differentials \(\rightarrow \mathcal{L}_1^* \otimes \mathcal{L}_2 \otimes \mathcal{O}(-p_0 - p_\infty)\)

so that this portion of the complex has cohomology \(= H^*(\mathbb{P}^1, \mathcal{O}(k))\) in any case.

So: if all the \(\mathbb{P}^1\)'s are punctured (have no area) \(\iff\) no smoothing on B side, so that only these "thin" Floer differentials occur, then we get

\[
\begin{align*}
H^* \text{hom}(L_1, L_2) & \cong H^*(\mathbb{P}^1, \mathcal{L}_1^* \otimes \mathcal{L}_2)
\end{align*}
\]

The hom generators correspond to monomials in the y word, in each \(\mathbb{P}^1\)

giving the \(\mathbb{P}^1\)'s finite area will deform Floer theory by allowing more bottom curves than just the thin ones, and this compared to deforming the curve \(\mathcal{C}_0\) to the smoothing described above.
Proposed rule for the Floer differential on $\text{hom}(l_1, l_2) = \text{CF}(l_1, l_2)$

= count "rigid" (0-dim! forties) holomorphic strips with boundary on $l_1, l_2$
connecting a deg-0 generator to a deg-1 generator; there can pass
through the vertices and behave as dictated by the chart data.

A possible holomorphic strip contributing to $\exists \in \text{hom}(A, B)$,
with input a and output b, coming into vertex as a "deg-1 strip"
(covers 0D almost once) and leaving
as a "deg-0 strip".

• A strip coming in a vertex with degree $d_1 \geq 1$ on leg labelled by
loc. word $P_1$ and going out with degree $d_2 \geq 0$ on leg labelled $P_2$
is allowed iff the coefficient of $P_2^{-d_2}$ in the power series expansion
of $P_1^{d_1}$ near the pole of $P_2$ is nonzero, and then it gets counted
with a multiplicity equal to this coefficient.

(The total weight of a strip that obeys the rule is:
$\exists \sim \text{area} \cdot (\text{holonomy of loc. syst. at boundary}) \cdot \text{TT(multiplicities at vertices)}$
crossed by the strip

$\exists_{1}$

$P_2 = \frac{1}{z}$

$P_3 = \frac{-1}{2+1}$

$d_1: 1$

$P_1 = z$

\begin{align*}
P_1 &= P_2^{-d_1} \\
\quad \text{if} \quad P_1 &= P_2^{-d_1} \\
\quad \text{if} \quad P_1 &= -1 - \frac{1}{P_3} \\
\quad d_1 &= (-1)^{d_1} \left(1 + dp_3^{-1} + (d_3) p_3^{-2} + \ldots \right) \\
\quad \text{while: strip coming in with degree } d_2 \text{ on leg } P_2 \text{ can either exit with degree } d_1 = d_2 \text{ on leg } P_1 \text{ or with any degree } d_3 \text{ on leg } P_3 \quad (P_2 = -\frac{P_3}{P_3 + 1} = -1 - P_3^{-2} - P_3^{-4} - \ldots)
\end{align*}
Example:

\[ \exists x_0 = 0 \quad \text{(two thin slits } x_0 \to y \text{ cancel each other)} \]
\[ \exists x_1 = \pm \text{ area } y \quad \text{(only traj. is as shown, } \frac{-1}{z^{2+1}} = \int \frac{1}{z} - \int \frac{1}{z^2} \ldots \text{ whereas } \frac{-1}{z^{2+1}} = \ldots \text{ has no contact term so if going into } \Theta \text{ leg must acquire degree } \geq 1, \text{ then keeps propagating around } \Theta \text{ and cannot ever end at } y \ldots \) \]
\[ \exists x_2 = \pm \text{ something } y \quad \text{(same: only } 1 \text{ trajecory, incoming degree } 2 \text{ and outgoing degree } 0 \text{ in leg } \Delta^{1/2} \) \]

Note: the asymmetry between \( \Theta \) and \( \Theta' \) is why we end up with \( H^1 \hom = 0 \) as expected!

This is consistent with \( \tilde{f} \) function on \( \Theta \) with pole order 1.

But \( \exists \) additional basis with one generator for each pole order \( \geq 2 \).

Can in fact get ring structure to match too! - other change of vars, get

\[ K[x,y]/y^2 = x^3 + ax + b \quad (\text{where } x \to x_2 + \ldots x_1, \ y \to x_3 + \ldots) \]

\( \mu^2 \) and higher products: similarly count helix polygons with boundary or suitably perturbed trivalent legs, graphs, with propagation along the path from inputs to output i.e. trees of discs attached together at the trivalent nodes! (+special case for output at a node).

With this understood,

Then (A. Efimov Kazakov in progress):

The Fubaya cut of a bordered config of \( C \)'s and \( P^1 \)'s defined in this way is
dar equivalent to about sheaves on the mirror curve constructed by giving \( (P^1, 3 \text{pts}) \)
and smoothing as provided.
Example of $x^2$ propagating:

propagation through a node, here degree 1 $u_1$
$\text{deg } u_1 \rightarrow \text{deg } 0 u_2$

$\text{multiplicity } = \text{expand } w_1 \rightarrow w_2$

$w_0 + w_1 w + w_2 w^2 + \ldots$

This is coefficient for $\text{deg } u_2 = 0$ $w_1 / \text{deg } u_2$

Special case: output at a node:

$\text{card } w_1 \text{ deg } r$
$\text{deg } 5 \text{ card } w_2$

$w_1^5 = \text{expand } w_2^5 = C + \sum_{k=1} a_k w^k + \sum_{l=1} a_l w^l$

This is the multiplicity.