The geometry of Lagrangian torus fibrations

Assume $\pi: (M^{2n}, \omega) \rightarrow B^n$ has Lagrangian fibers over regular values of $\pi$.

A proper map $F_b = \pi^{'-1}(b)$.

- Flux gives a local chart $U_{b_0} = \text{nil of } b_0 \in B \xrightarrow{\phi} \mathbb{H}^{4}(F_{b_0}, \mathbb{R})$.

  Abstr: nil of $F_{b_0} \cong \text{nil of zero section in } T^*F_{b_0}$, and for $b \text{ close to } b_0$,

  $F_b = \text{graph of a closed 1-form } \omega_b$. $\phi(b) = [\omega_b] \in \mathbb{H}^4(F_{b_0}, \mathbb{R})$.

  Explicit: $\forall \zeta \in \mathcal{H}^1(F_{b_0})$, $\langle \phi(b), \zeta \rangle = \text{symplectic area of cylinder } \gamma$.

  (index of choice by Stokes)

- This map is always a local embedding: injectivity: if $\phi(b) = \phi(b') \Rightarrow \omega_b - \omega_{b'} = dF$, but $F \cap F_b = \emptyset \Rightarrow F$ has no critical points, contradiction.

  Similarly, different at zero: $\forall \zeta \in \mathcal{T}_{F_{b_0}} B \Rightarrow \zeta^! \text{ any lift, } d\phi(b) = [-\zeta \wedge \omega]_{F_{b_0}} \in \mathbb{H}^4(F_{b_0}, \mathbb{R})$ can be exact by the same argument (the closed 1-form $\omega \wedge \zeta$ has no zeros since $\zeta^!$ number tangent to $F_{b_0}$).

- So $\dim \mathcal{H}^4(F_{b_0}) \geq n$, and $T^*F_{b_0}$ has nd (e.g. $T^*F_{b_0} \cong T^*_b B \times F_{b_0}$).

  Case of interest: regular fibers are tori. Then $\phi$ a local chart $B \cong U_{b_0} \rightarrow \mathbb{R}^n$.

- Then we have an integer affine structure on $B$ (or on regular locus $B^\circ \subset B$).

  i.e. local charts into $\mathbb{R}^n$, with transition functions in affine group $GL(n, \mathbb{Z}) \ltimes \mathbb{R}^n$ (translations = change reference fiber, $GL(n, \mathbb{Z}) = \text{change basis of } H_1(F)$).

  Equivalently, affine str = data of a family of lattices $\Lambda = \mathcal{T}B \subset \mathcal{T}B$.

* Connection to completely integrable systems = n commuting Hamiltonians $f_1, \ldots, f_n : M \rightarrow \mathbb{R}$ with $\{f_i, f_j\} = 0$; $d f_1, \ldots, d f_n$ linearly independent over regular values of $\pi = (f_1, \ldots, f_n): M \rightarrow \mathbb{R}^n$.

  $X_i = X_{f_i}$ satisfy $d f_j(X_i) = \omega(X_i, X_j) = 0$ i.e. the $i$th $X_i$ are tangent to fibers of $f$.

  At a regular point, $X_i$ linearly indep $\Rightarrow$ span target space to the fiber, which is hence Lagrangian.

So $f$ is a Lagrangian fibration away from critical points.

Example moment map $\mu : M \rightarrow \mathbb{R}^n$ on a toric sympl manifold. In this case the affine structure on the base = $\mu(M) = \Delta CR^n$ exactly given by std coords. (p. 102)

Example $\pi : M \rightarrow B$ Lagr fibration, $\phi : U_{b_0} \rightarrow \mathbb{R}^n$ local coord = $f = \phi^{'-1}$ locally integrable system!

Corollary: We could do this for any local coord on $\phi$, but advantage to use affine coord/flux chart...
A priori the flows of $X_i = X_i^f$, in a locally integrable system, have no requirement to be periodic. Certainly these are models of critical pts of $f$ whose periodicity is improbable. Yet, over the regular part, this is basically automatic.

Announce coordinates: $f = (f_1...f_n)$ loc. integrable system, near a regular value $b_0$.
Assume $F_0 = f^{-1}(b_0)$ compact closed (eg. $\Lambda$ compact or $f$ proper), by above it's Lagrangian.
Along $F_0$, $(dX_1...dX_n)$ define basis of $\frac{T^*F_0}{T^*F_0} \cong \frac{T^*X_0}{\omega}$.

Working in a $\epsilon$-affine nbhd. we can identify locally $X$ with $T^*F_0$, $F_0 = \epsilon$-lab action nearby $f^{-1}(b) = \text{graph}$.

Then $\exists$ closed 1-forms $\alpha_1...\alpha_n$ on $F_0$ st.
$$f^{-1}(b_0 + (\lambda_1...\lambda_n)) = \text{graph}(\sum \lambda_i \alpha_i + O(\epsilon^2))$$

pickwise $\alpha_1...\alpha_n$ basis of $\frac{T^*F_0}{T^*F_0} \cong \frac{T^*X_0}{\omega}$

and $dX_i(\alpha_j) = \delta_i^j$.

ie. $\{\alpha_j\}$ is dual basis to $\{X_i\}$.

Now, $\alpha_j$ closed = on universal cover $\widetilde{F_0}$, lift of $\alpha_j$ is exact = $d\Theta_j$.

and flow of $X_j$ lifts with $d\Theta_j(X_i) = \delta_j^i$, ie. $\Theta_j: \widetilde{F_0} \to \mathbb{R}^n$, $X_j = \frac{\partial}{\partial \Theta_j}$.

Since $X_j$ pointwise linearly indep, $(\Theta_1...\Theta_n): \widetilde{F_0} \to \mathbb{R}^n$ is a local diffeo

hence a covering map hence a global diffeo. However deck transformations of $\widetilde{F_0} \to F_0$ act by translations (poinc of the 1-forms $\alpha_j$) $\Rightarrow F_0 = \mathbb{R}^n/\text{lattice} \cong T^n$.

(Arnold-Liouville thm.: the compact orj fibers of a ci-system are tori.)

* If we pick our coords. $(f_1...f_n)$ on the base to be the affine ones, ie.

$[\alpha_1]...[\alpha_n]$ is a basis of $H^1(F_0, \mathbb{Z})$ then the lattice of periods is $\mathbb{Z}^n \subset \mathbb{R}^n$,

ie. we've identified $\Theta_j$ = usual coordinates on base $F_0 = T^n = (\mathbb{R}/\mathbb{Z})^n$,

and $X_j = \partial/\partial \Theta_j$. Thus: Ham. flows are periodic! Locally a Ham. $T^n$-action!

* The angle coords $\Theta_j$ on $F_0$ (or any regular fibertr) are only defined up to an additive constant. Normalize these so that $\Theta = 0$ is locally a Lagrangian section of $f$.

(eg. some particular fiber of $T^*F_0$ is local model) (then by action of $X_i = \frac{\partial}{\partial \Theta_i}$, so an all the other sections $\Theta = \text{const.}$) Then $\omega = \sum_{ij} \delta_{ij} d\Theta_i \wedge d\Theta_j$, and by const. $X_i = \frac{\partial}{\partial \Theta_i}$ so $\Rightarrow \omega = \delta_{ij}$.

$\omega = \sum_{ij} dX_i \wedge d\Theta_j$. 

$\Rightarrow$ $\omega_{ij} = \delta_{ij}$.
Hence, near \( F_0 \), \( \pi : M \to B \) looks like \((T^*B/T^*B^2, \omega_0 = \Sigma d\phi_i \wedge d\theta_i)\).

Affine chart \((c_i, -c_i)\), \(\text{span}_{\mathbb{R}} \big( \frac{\partial}{\partial \theta_i} \big) = TB^2\)

Dual chart \((\theta_1, \theta_n)\) on \(T^*B\), and integer lattice \(TB^2\).

(Globally, there are obstructions to the existence of LLP sections.)

The local picture is boring but, as in the case of \(\text{toric} \) manifolds, it is the critical points of \(\pi\) that make things interesting.

Ex: a toric manifold, \(\mu = \text{moment map}\).

**Eg:** \((\mathbb{C}P^2, \omega)\), \(B = \mathbb{C}P^1\)

\[\mu = \left( \frac{lx^2 - ly^2}{lx^2 + ly^2 + |z|^2}, \frac{|xy - cz|^2}{lx^2 + ly^2 + |z|^2} \right)\]

\(\mu\) is moment map for \(S^1\)-action

\(\text{rotating } x \& y \text{ in opposite directions}\)

\(f = \text{not normalized to affine chart.}\)

In reduced space \(\mu^{-1}(\lambda)/S^1 \simeq \mathbb{C}P^1\)

levels are a family of curves "centered at" \(\frac{xy}{z^2} = \lambda\)

The normalized coord. = amount of area enclosed.

(The total area at level \(\lambda \) is \(\frac{1-|\lambda|^2}{2}\).)

Analogous in \(\mathbb{R}^2\) is \((lx^2 - ly^2, |xy-cz|)\)

\((x,y)\)

\(\frac{1}{2}\)

\(xy\)

\(c\)

Only singularity is at origin \(f = (0, 1lc)\). The sing. fiber = \(\mathbb{C}^0\)

"Whitney sphere" = immersed \(S^2 \times \mathbb{S}^2\) with one double point.

The nearby smooth fibers \(f^{-1}(0, r)\) for \(r \neq 1lc\), are product-type resp. chevron-type \(\mathbb{S}^2\)-tori; for \(r \approx 1lc\), they are hom. to the outcome of Polterovich surgery at the double point of the sing. fiber (in one direction or the other).

The affine structure on the base \(B\) looks like

At the branch cut, monodromy \(\sim \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\)

Reflecting monodromy on \(H^1(\text{fiber})\) (shear/\(S^1\)-orbit dir.) \(-1\)

\(\frac{1}{2}\)

\((-1, -1)\)

\((-\infty, 1)\)}
For $c=0$ this reverts to the toric picture ($\mathbf{CP}^1$ product torus when $c=0$).

For $c=\frac{1}{2}$ this becomes another interesting $\mathbb{R}$ system on $\mathbb{CP}^2$:

\[
\begin{align*}
|xy-\frac{1}{2}z^2| &\leq \frac{1}{2} \quad \text{equality case: normalize so } z \in \mathbb{R}, \text{ then equality iff } y = -\overline{x}.
\end{align*}
\]

This for $c=\frac{1}{2}$, $\text{pt}^{-1}(0,\frac{1}{2}) \cong \mathbb{RP}^2$ (fund pt set of $\mathbb{C}$ conj.) (0,1) \rightarrow (-\overline{y} : \overline{x} : \overline{z})

The base looks like

\[
\begin{align*}
\text{pt} \quad \mapsto \quad (\frac{1}{2}, 0) \quad \mapsto \quad 2 \\
(0,1) \quad \mapsto \quad \text{pt} \\
(0,-1) \quad \mapsto \quad \text{pt}
\end{align*}
\]

There is a calculus of such deformations of integrable systems in dim 2 (Smythman).

"Nodal slide" as $c$ varies

Starting from toric $\mathbb{CP}^2$, get

Then successive nodal slides give

The set of shapes we can reach is all

\[
\begin{align*}
(a^2, b^2, c^2) \text{ s.t. } a^2 + b^2 + c^2 = 3abc
\end{align*}
\]

"Nodal slide" $c^2 \rightarrow b^2 \rightarrow a^2 \rightarrow c^2$ with $c' = 3ab - c$; "mutation".

The fiber at a well-chosen point (affine basecenter) of the base is a monotone Lagrangian torus $T_{a^2, b^2, c^2} \subset \mathbb{CP}^2$. They are all different. (R. Vianna 2014)

\[
\begin{align*}
T_{1,1,1} &\quad = \text{monotone product torus bounds 3 families of min-area laminations} \\
T_{1,1,4} &\quad = \text{Chekanov} \quad 4 \text{ families (nodal multiplicity 5)} \\
T_{1,4,25} &\quad = \text{Vienna} \quad 10 \quad 41
\end{align*}
\]