Course overview

* Introduction
  * Main goal: symplectic manifolds, Lagrangian submanifolds & Lagrangian Floer homology.

Today: overview of the course (at a rapid pace, we'll return to it more gently).

1. **Symplectic manifolds**: \((M^{2n}, \omega)\), smooth manifold, \(\omega \in \Omega^2(M)\),
   - non-degenerate: \(\frac{\partial \omega}{\partial n} \neq 0\) pointwise \(\Rightarrow\) vol. form, orientation
   - closed: \(d\omega = 0\).

   Examples:
   - \(M = \text{oriented surface}\), \(\omega = \text{area form}\)
   - \(M = \mathbb{R}^{2n}\), \(\omega = \sum dx_i \wedge dy_i\)
   - \(M = T^*\mathbb{N}\), \(\omega = d\lambda\), \(\lambda = p dq\) Liouville form
   - \(M = \mathbb{C}P^n\), complex projective space...

2. **Then (Darboux)**: \(\forall p \in M \exists \text{bd. & local coords } (x_1, x_2) \text{ in which } \omega = \sum dx_i \wedge dx_i\)

   No local invariants! Obvious global invariant: \([\omega] \in H^2(M, \mathbb{R})\)

3. **Then (Moser)**: \(M \text{ compact, } (\omega_t)_{t \in [0, 1]} \text{ symplectic forms with } [\omega_t] \in H^2(M, \mathbb{R}) \text{ indep of } t\n\)

   \(\Rightarrow\) \(\exists\) isomorphism \(\varphi_t \in \text{Diff}(M)\) s.t. \(\varphi_t^*\omega_t = \omega_0\), in particular

   \((M, \omega_0) \overset{\varphi_1}{\sim} (M, \omega_1)\).

Moreover, the group of symplectic morphisms \(\text{Symp}(\mathbb{R}^n, \omega)\) is very large!

\(\text{Symp} \supset \text{Ham}(\mathbb{R}^n)\) Hamiltonian diffeos = flow of (time dependent) \(\text{Ham. v.f.}\)

\(H \in C^\infty(M, \mathbb{R}) \Rightarrow \exists! \text{ v.f. } X_H \text{ s.t. } \omega(X_H, \cdot) = -dH.\)

Its flow preserves \(\omega\):

\(L_{X_H} \omega = \frac{d}{dt}(\varphi_t^*\omega + c_H d\omega) = 0.\)

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2. **Lagrangian submanifolds**: \(L^n \subset (M^{2n}, \omega)\) s.t. \(\omega|_L = 0\)

Examples:
- \(\mathbb{R}^n_x \subset (\mathbb{R}^{2n}(x, y), \omega_0)\)
  - any simple closed curve on a surface
  - in \(T^*\mathbb{N}\), the zero section, cotangent fibers...
  - \(T S^*(\mathbb{R}) \subset \mathbb{R}^{2n}\) or \(T^*\mathbb{C}P^n\). Not generally,

\(T^n\) acts in toric symplectic (i.e. Hamiltonian \(T^n\)-action on \(\mathbb{R}^{2n}\))

(ie. level sets of moment map \(\mu : (p, \ldots, p')\) Ham's generating functions)
Observe: $T_{L}^{\perp} = TL$, so $\omega$ induces an isomorphism $NL = TM_{CL}/TL \to TL$

In fact, Weinstein's theorem: a neighborhood of $L$ in $M$ is symplectomorphic to a tub of the zero section in $T^{*}L$.

Deformation of subfields $\leftrightarrow$ section of normal bundle

In $\text{Lap. case}$, a deformation is Lagrangian isotopy iff $\omega$ is graph of closed 1-form

Ham. isotopy $\leftrightarrow$ exact 1-form.

Ex. $L = \{x \in R^{3} \mid x_{1} = 0\}$, $\omega = dx_{2} \\
\\nS_{\omega} = \text{flux of the deformation } \in \mathbb{R}$

(Incidentally, $H^{1}(L, \mathbb{R})$ measures the difference between Ham. & Lag. isotopy.)

First 3 weeks: go over this & other classical symplectic geometry.

3. Classification questions: what kinds of Lag. subfields (up to Lag. / Ham. isotopy) does $M$ contain?

- Ex. on a surface, closed Lag. 2-forms, Ham. iso $\iff$ no area swept.

- Arnold nearby Lag. conjecture: $L \subset T^{*}N$ closed exact Lag. submanifold

$\Rightarrow L$ has an exact Lagrangian submanifold.

Exact means: $(T^{*}N, \omega = d(\lambda))$ exact symplectic form

$L$ Lag. $\iff \lambda|_{L}$ closed

Say $L$ exact Lag. if $\lambda|_{L}$ is exact.

Eg. graph(orthogonal) Lag. $\iff \alpha$ closed exact (then Ham. iso to zero set).

Proved for only $T^{*}S^{1}$ (easy), $T^{*}S^{2}$ (Hind 2003), $T^{*}T^{2}$ (DimitryA Rizell, Goodman, Hryniewicz 2014)

though... Thm. (Abreu-Kragh 2016): (after Fukaya-Seidel-Smith, Nadler, ...)

$L \subset T^{*}N$ closed exact Lagr. $\Rightarrow \pi_{2}|_{L}: L \to N$ is a simple homotopy equivalence.

- in $R^{6}$: $L \subset R^{6}$ closed Lagr. $\Rightarrow \begin{cases} \text{if orientable, } L \cong T^{2} \\
\text{if not, } \chi(L) < 0 \text{ divisible by } 4. \text{ (Givental)} \\
\text{case of Klein bottle excluded by Nemirovskii 2006.} \end{cases}$

All known Lagr. tori in $R^{6}$ are Ham. iso to

$\{\text{product tori } S^{1}(r_{1}) \times S^{1}(r_{2}) \}

\text{or Chekanov torus } T_{ch}(r).

\Rightarrow \text{1990 } \exists T_{C}(R^{6}, \omega_{0}) \text{ not } \leq \text{ any product torus.}

\text{Conj. no others.}
Note: Gromov: [\\textbf{closed exact Lagrangians in } \mathbb{R}^{2n}] \text{. Next:}

- exact Lagrangian immersions: must satisfy an h-principle (Gromov)
  in particular \( L \subset \mathbb{R}^{2n} \text{ iff } TL \otimes \mathbb{C} \text{ is a trivial vector bundle.} \)

- monotone embedded Lagrangians: given a disc \( u: (\mathbb{D}^2, \partial) \to (N, L) \),
  \( \text{area } \int u^* w = \int \mu(u) \)
  \( \text{Red content } \mu \text{ almost class } \geq 0 \)
  \( \text{of } TL \text{ around } \partial \mathbb{D}^2 \).

In \( \mathbb{R}^6 \):
- Fukaya: \( L \text{ closed monotone twisted } \mathcal{L} \subset \mathbb{R}^6 \Leftrightarrow L \circ S^1 \times S^1 \).
- Ekholm, Eliashberg, Ng, Smith 2013: \( \mathbb{R}^3 \text{ closed twisted, } \exists \text{ exact Lag. immersion } N \to \mathbb{R}^6 \text{ with just one double point.} \)
  \( \text{Using Lap. surgery, } \Rightarrow \exists \text{ Lag. embedding } N \# (S^1 \times S^2) \subset \mathbb{R}^6 \).

Recent developments on Lag. tori: \( \text{Thur (Vienna 2014) } \| \mathbb{D}^2 \Rightarrow \text{ only many different } \)
\( \text{monotone Lag. tori.} \)

\( \text{Thur (2016) } \| \mathbb{R}^6 \Rightarrow \text{ (Similarly high dim.).} \)

(known once all Lag. isotopic to products).

\( \Leftrightarrow \) whereas: Dinkelbach & Evans point out \( \exists \text{ monotone } T^4 \subset \mathbb{R}^8 \)
  \( \text{but isn’t smoothly isotopic to a product torus.} \)

\( \text{Plan:} \) won’t get into all these results (some of which are quite technical) but study a set of tools
  \( \text{& explain some of the applications.} \)

\( \text{4. key tool to study Lagrangians: } \text{holomorphic discs.} \)

- every sympl. ndd admits a compatible almost-com. structure \( J \in \text{End}(TM), J^2 = -1 \)
  & set of chiral is contractible

- pseudo-holomorphic curves: \( u: (\Sigma, j) \to (M, \omega) \text{ s.t. } \bar{\partial}_j u = 0 \text{ i.e. } du \circ j = J \circ du. \)

- the deformation theory of J-curves is governed by a Fredholm operator
  \( (\bar{\partial}_j \text{ operator on sections of } u^* TM) \text{ so eqn. finit-dim. space of solutions,}
  \text{count when 0-dim!} \)
need to understand transversality & encompassness properties of
module spaces. key phenomenon: breaking & controlling it by
symplectic area

$S^1$ bundle is transverse that can be normalized,

\[ \bigcirc \quad \xrightarrow{u_n} \quad \bigcirc \quad \xrightarrow{n \to \infty} \bigcirc \]

**PLAN**

Spend a few weeks on these foundations & then
defining invariants based on this.

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(5) To study Lag's & their intersection properties:

Lag's Floer homology

\[ \text{HF}(L_1, L_2) = H^*(\text{CF}(L_1, L_2), \mathbb{Z}) \]

\[ \text{CF} = \text{vector space generated over some coefficient ring} \]

\[ \text{by } L_1 \cap L_2 \quad \text{(symplectic) \quad \text{(assume } \Lambda \text{)}} \]

\[ \langle \mathfrak{p}, q \rangle = \text{(weighted) count of } \{ \text{c} : \mathbb{R}^n \to q, \int_0^1 \mathfrak{p} = 0 \}/R-bnd. \]

**Floer:**

- If \( L_i \) don't bound any holo. disc (eg. exact case) then
  - \( \mathfrak{g} = 0 \)
  - HF is invariant under Hamiltonian isotopies
  - \( \text{HF}(L_1, L_2) = H^*(L) \)

**Contrary:** \( \#(L \cap \mathcal{P}(L)) \geq \text{dim } H^*(L) \) whenever \( \mathcal{P} \in \text{Ham} \& \mathcal{P}(L) \cap L \).

(Compare for small isotopies, \( \mathcal{P}(L) = \text{graph}(dF), \quad F \in C^0(L) \) Morse
and the ineq. \equiv Morse inequality for \#crit(F)).

**Ex:**

\[ \text{Ex. } \quad \begin{array}{c}
\pin L_1, \quad \mathfrak{p} = q - q = 0 \Rightarrow \text{HF}(L_1, L_2) = H^*(S^1)
\end{array} \]

**Context:**

\[ \begin{array}{c}
\text{Counter-ex. } \quad \begin{array}{c}
\text{L}_2 \quad \text{Harm.}
\end{array}
\end{array} \]

- To study monotone Lagrangians: study counts of holomorphic discs with \( \mathfrak{g} \) on \( L \)
  - (generally make sure of Floer theory when \( \exists \) disc)

  \( \text{L} \) obstruction in Floer theory if \( \exists \) disc of Nahm index \( \leq 2 \)

  \( \Rightarrow \) counts of Nahm \( \leq 2 \) discs give invariants of monotone Lag's up to isotopy,

  \( \text{used to distinguish Chekanov tori \& many examples.} \)
Fukaya category: organize all Lag's $\subset (M,\omega)$ for which $\text{Lag} \cdot \text{Floer} \text{ theory}$ is well-defined & their intersection properties.

- **Obj:** $L \subset (M,\omega)$
- **Morphisms:** $\text{CF}(L_1,L_2)$, $\mathcal{D} = \mu^1$.
- **Composition:** $\mu^2: \text{CF}(L_1,L_2) \circ \text{CF}(L_0,L_1) \to \text{CF}(L_0,L_2)$ Floer product
  + higher operations $\mu^k$

Point: $\to$ easier to understand Lag's up to Floer-theoretic iso. than up to Ham. iso.
  $\to$ thanks to homological algebra, it may be enough to understand Floer theory for a finite set of generators of $\mathcal{F}(M)$, exposing all others in terms of these. For instance, when $M = T^n N$,
  $\to$ compact exact Fukaya cat. is generated by zero section
  $\to$ wrapped Fukaya cat. (noncompact exact Lag's with spoke perturbation at infinity) is generated by cotangent fiber.

* Besides symplectic geometry, Fukaya categories also play a key role in
  $\to$ mirror symmetry: Fukaya cat. of $M$ $\leftrightarrow$ derived cat. of coherent sheaves on $M^\vee$
  $\to$ low-dim. top: Fukaya cat. of certain configuration spaces are where various invariants of 3-manifolds, knots & links live naturally.
  E.g. Heegaard-Floer homology $= \text{Floer homology in } \text{Sym}^3(I_g)$
  but also Khovanov homology (Seidel-Smith/Abouzaid).