1. (22.2) (a) Assume \( p : X \to Y \) is continuous, and \( f : Y \to X \) is a continuous map such that \( p \circ f = \text{id}_Y \). Then \( p \) is surjective (since for any \( y \in Y \), \( y = p(f(y)) \in p(X) \)), and continuous so if \( U \subset Y \) is open then \( p^{-1}(U) \subset X \) is open. To show the converse implication (if \( p^{-1}(U) \subset X \) is open then \( U \subset Y \) is open), we note that \( U = (p \circ f)^{-1}(U) = f^{-1}(p^{-1}(U)) \), and \( f \) is continuous, so if \( p^{-1}(U) \subset X \) is open then \( f^{-1}(p^{-1}(U)) = U \subset Y \) is open.

(b) Assume \( A \subset X \) and \( r : X \to A \) is a retraction. Let \( i : A \to X \) be the inclusion map. Then \( i \) and \( r \) are continuous, and \( r \circ i = \text{id}_A \). By the result of (a), it follows that \( r \) is a quotient map.

2. The quotient \( X/ \sim \) “looks like” \( \mathbb{R} \), except there are two different points living at the origin: for \( x \in \mathbb{R} - \{0\} \) we have a single equivalence class \([x \times 1] = [x \times 2] \), while \([0 \times 1]\) and \([0 \times 2]\) are two different equivalence classes. Write \( p : X \to X/ \sim \) for the quotient map. Let \( U_1 \) be a neighborhood of \([0 \times 1]\) and \( U_2 \) a neighborhood of \([0 \times 2]\) in the quotient topology. By definition, \( U_1 \) being open in the quotient topology means that \( p^{-1}(U_1) \) is open in \( X = \mathbb{R} \times \{1, 2\} \). Since \([0 \times 1] \in U_1 \), \( 0 \times 1 \in p^{-1}(U_1) \), and hence there exists \( \epsilon_1 > 0 \) such that \((-\epsilon_1, \epsilon_1) \times \{1\} \subset p^{-1}(U_1)\). Similarly, there exists \( \epsilon_2 > 0 \) such that \((-\epsilon_2, \epsilon_2) \times \{2\} \subset p^{-1}(U_2)\). Now, take \( x \in \mathbb{R} \) such that \( 0 < x < \min(\epsilon_1, \epsilon_2) \). Then \( x \times 1 \in p^{-1}(U_1) \) so \( p(x \times 1) = [x \times 1] \in U_1 \), and \( x \times 2 \in p^{-1}(U_2) \) so \( p(x \times 2) = [x \times 2] \in U_2 \). However the latter two elements of \( X/ \sim \) are equal to each other since \( x \neq 0 \), so we find that \( U_1 \cap U_2 \supseteq [0 \times 1] = [x \times 2] \) is non-empty. So \([0 \times 1]\) and \([0 \times 2]\) do not have disjoint neighborhoods, hence \( X/ \sim \) is not Hausdorff.

3. (a) **Solution 1** (comparing the quotient topologies “by hand”)

Every line through the origin in \( \mathbb{R}^{n+1} \) contains exactly two points on the unit sphere \( S^n \), which are antipodal to each other. Namely, the two points of \( S^n \) in the equivalence class of \( x \in X \) are \( \pm \frac{x}{|x|} \), and they form an equivalence class for the equivalence relation on \( S^n \). Thus, the sets of equivalence classes for the equivalence relations on \( X \) and on \( S^n \) are naturally identified with each other (by mapping the equivalence class of \( x \in X \) to that of \( \frac{x}{|x|} \in S^n \)).

We denote by \( p : X \to \mathbb{RP}^n \) and \( q : S^n \to \mathbb{RP}^n \) the two quotient maps, and note that \( q = p_{|S^n} = p \circ i \) where \( i : S^n \to \mathbb{RP}^{n+1} - \{0\} \) is the inclusion. Moreover, denote by \( r : X \to S^n \) the projection \( r(x) = x/|x| \), which is a retraction. Since \( r \) preserves each equivalence class, we also have \( p = q \circ r \).

To show that the two quotient topologies coincide, we then need to show that for any subset \( U \subset \mathbb{RP}^n \), \( q^{-1}(U) \) is open in \( S^n \) if and only if \( p^{-1}(U) \) is open in \( X \). (By definition of the quotient topology, this says: \( U \) is open in \( S^n/ \sim \) if and only if \( U \) is open in \( X/ \sim \)). For this, note that \( q^{-1}(U) = i^{-1}(p^{-1}(U)) = p^{-1}(U) \cap S^n \). So if \( p^{-1}(U) \) is open in \( X \), then \( q^{-1}(U) \) is open in \( S^n \) (by continuity of \( i \), or by definition of the subspace topology). Conversely, \( p = q \circ r \), so \( p^{-1}(U) = r^{-1}(q^{-1}(U)) \), and the continuity of \( r \) implies that if \( q^{-1}(U) \) is open in \( S^n \) then \( p^{-1}(U) \) is open in \( X \).

**Solution 2** (using the characterization of continuous maps from quotient spaces)

Denote by \( p : X \to X/ \sim = \mathbb{RP}^n \) and \( q : S^n \to S^n/ \sim \) the two quotient maps; by \( i : S^n \to X \) the inclusion, and \( r : X \to S^n \) the projection \( r(x) = x/|x| \) (which is a retraction).

The map \( p \circ i : S^n \to X/ \sim \) is continuous, and compatible with the equivalence relation on \( S^n \), since \( p \circ i(x) = p \circ i(-x) \), so by Theorem 22.2 (see also Thm. on p.3 of lecture notes for Nov. 4), it induces a continuous map \( i : S^n/ \sim \to X/ \sim \), defined by \( i([x]) = [i(x)] \). Conversely, the map \( q \circ r : X \to S^n/ \sim \) is continuous, and compatible with the equivalence relation on \( X \); indeed,
\( r(\alpha x) = \pm r(x) \), so \( q \circ r(\alpha x) = q \circ r(x) \). Hence by Theorem 22.2 it induces a continuous map \( \tilde{r} : X/\sim \to S^n/\sim \), defined by \( \tilde{r}(x) = [r(x)] \). Summarizing, we have a diagram

\[
\begin{array}{ccc}
X & \xrightarrow{r} & S^n \\
\downarrow \ i \ & & \downarrow \ q \\
X/\sim & \xrightarrow{\tilde{r}} & S^n/\sim 
\end{array}
\]

where all the maps are continuous. We claim that \( \tilde{r} \) and \( \tilde{i} \) are inverse maps. Indeed, for \( x \in S^n \), \( \tilde{r} \circ \tilde{i}(x) = \tilde{r}(i(x)) = [r(i(x))] = [x] \), and for \( x \in \mathbb{R}^{n+1} - \{0\} \), \( \tilde{i} \circ \tilde{r}(x) = \tilde{i}(\tilde{r}(x)) = [i(r(x))] = [x/|x|] = [x] \). So \( \tilde{r} \) and \( \tilde{i} \) are continuous bijections with continuous inverses, hence homeomorphisms.

(b) Let \( [x_0] \in \mathbb{RP}^n \) be the equivalence class of a pair of antipodal points \( \pm x_0 \in S^n \). Let \( V_+ \) be a small neighborhood of \( x_0 \) in \( S^n \), for example, the intersection of \( S^n \) with the ball \( B(x_0, \frac{1}{2}) \subset \mathbb{R}^{n+1} \), and let \( V_- \) be the corresponding neighborhood of \( -x_0 \), \( V_- = -V_+ = \{ -x \ | \ x \in V_+ \} \). Then \( q(V_+) = q(V_-) = U \) is a neighborhood of \( [x_0] \) in \( \mathbb{RP}^n \), with \( q^{-1}(U) = V_+ \cup V_- \). Having chosen \( V_+ \) small enough as above ensures that \( V_+ \) and \( V_- \) are disjoint (since the balls of radius \( \frac{1}{2} \) centered at \( \pm x_0 \) are disjoint). Moreover, we claim that the restriction of \( q(V_+) : V_+ \to U \) is a homeomorphism. Indeed, a subset \( W \subset U \) is open if and only if \( q^{-1}(W) \) is open in \( q^{-1}(U) = V_+ \cup V_- \), which is equivalent to \( q^{-1}(W) \cap V_\pm \) both being open. Therefore, the neighborhood \( U \) of \( [x_0] \) is evenly covered by \( q \) (with the two slices being \( V_\pm \)). This is true for every point of \( \mathbb{RP}^n \), and hence \( q : S^n \to \mathbb{RP}^n \) is a (two-sheeted) covering map.

4. (55.1) Let \( A \subset B^2 \) be a retract, with \( r : B^2 \to A \) the retraction, and \( i : A \to B^2 \) the inclusion. Given a continuous map \( f : A \to A \), the composition \( F = i \circ f \circ r \) is a continuous map from \( B^2 \) to itself, with \( F(B^2) \subset A \), and \( F|_A = f \). By the Brouwer fixed point theorem, \( F \) has a fixed point, i.e. there exists \( x \in B^2 \) such that \( F(x) = x \). However, since \( F \) takes values in \( A \), this implies that in fact \( x \in A \), and then \( f(x) = F(x) = x \). So \( x \) is a fixed point of \( f \).

5. (55.2) If \( h : S^1 \to S^1 \) is null-homotopic, then by Lemma 55.3 it extends to a continuous map \( h : B^2 \to S^1 \). The composition of \( h \) with the inclusion \( i \) of \( S^1 \) into \( B^2 \) is a continuous map from \( B^2 \) to itself, so by the Brouwer fixed point theorem, there exists \( x \in B^2 \) such that \( i(h(x)) = x \). However, this equality implies that \( x \in S^1 \), so in fact there exists \( x \in S^1 \) such that \( k(x) = h(x) = x \), and we conclude that \( h \) has a fixed point.

Denote by \( \alpha : S^1 \to S^1 \) the antipodal map \( \alpha(x) = -x \). If \( h : S^1 \to S^1 \) is nullhomotopic, then so is \( \alpha \circ h \) (namely, if \( H \) is a homotopy from \( h \) to a constant map then \( \alpha \circ H \) is a homotopy from \( \alpha \circ h \) to a constant map). By the previous result, \( \alpha \circ h \) has a fixed point, so there exists \( x \in S^1 \) such that \( \alpha(h(x)) = x \), i.e. \( h(x) = -x \).

Remark: one can in fact show that every continuous map from \( S^1 \) to itself with no fixed point is homotopic to the identity map, and so is every continuous map such that \( h(x) \neq -x \) for all \( x \in S^1 \). So the desired conclusion holds more generally whenever \( h \) is not homotopic to the identity map. To prove the claim: assume \( h(x) \neq -x \) for all \( x \), then the straight line segment from \( h(x) \) to \( x \) does not pass through the origin, and we can define \( H : S^1 \times I \to S^1 \) by \( H(x,t) = r((1-t)h(x)+tx) \), where \( r : \mathbb{R}^2 - \{0\} \to S^1 \) is the retraction \( r(y) = y/|y| \). This gives a homotopy from \( h \) to the identity map. Similarly, assume \( h(x) \neq x \) for all \( x \), then by the same argument \( h \) is homotopic to the antipodal map \( \alpha(x) = -x \), which is in turn homotopic to the identity map (via rotations of angles varying continuously from 0 to \( \pi \)).
6. (57.2) As per the hint, we note that the complement of a point in the 2-sphere, $S^2 - \{p\}$, is homeomorphic to $\mathbb{R}^2$. For example, such a homeomorphism can be obtained by stereographic projection, mapping $S^2 - \{p\}$ onto the tangent plane $P$ to $S^2$ at $-p$, by mapping each point $x \in S^2 - \{p\}$ to the point where the line through $p$ and $x$ intersects the plane $P$; the inverse homeomorphism mapping each point $y \in P$ to the point of $S^2 - \{p\}$ where the line through $p$ and $y$ intersects $S^2$ (other than $p$). (Draw a picture!) (Optional: write formulas for the stereographic projection of $S^2 - \{(0,0,1)\}$ onto the tangent plane at $(0,0,-1)$, namely $P = \{(x,y,-1)\}$, and its inverse, to convince yourself that these maps are continuous).

Now, assume $g : S^2 \rightarrow S^2$ is continuous and not surjective, so $g(S^2) \subset S^2 - \{p\}$ for some point $p$. Denoting by $h : S^2 - \{p\} \rightarrow \mathbb{R}^2$ a homeomorphism, the composition $h \circ g$ is a continuous map from $S^2$ to $\mathbb{R}^2$, so by the Borsuk-Ulam theorem there exists $x \in S^2$ such that $h(g(-x)) = h(g(x))$. Applying the inverse homeomorphism $h^{-1}$ to both sides, this implies that $g(-x) = g(x)$. Conversely, if $g(-x) \neq g(x)$ for all $x \in S^2$ then $g$ must be surjective.

7. (a) Recall the Borsuk-Ulam theorem for $S^3$: if $f : S^1 \rightarrow \mathbb{R}$ is a continuous function, then there exists $x \in S^1$ such that $f(-x) = f(x)$. (The proof is elementary, see HW4 Problem 4 = 24.2).

Now, assume $S^1 = A \cup B$ where $A$ is closed (we don’t in fact need $B$ to be closed). The function $f : S^1 \rightarrow \mathbb{R}$, $f(x) = d(x, A)$ is continuous, so there exists $x \in S^1$ such that $f(x) = f(-x)$. If $f(x) = f(-x) = 0$, then so $A$ contains a pair of antipodal points. Otherwise, if $f(x) = f(-x) > 0$, then we conclude that neither $x$ nor $-x$ belong to $A$, hence they both lie in $B$.

(b) Assume $S^1 = A \cup B \cup C$, where $A$ and $B$ are closed. Define a continuous function $f : S^2 \rightarrow \mathbb{R}^2$ by $f(x) = (d(x, A), d(x, B))$. By the Borsuk-Ulam theorem, there exists a pair of antipodal points $\pm x$ such that $f(x) = f(-x)$, so $d(x, A) = d(-x, A)$ and $d(x, B) = d(-x, B)$. If $d(x, A) = d(-x, A) = 0$ then we conclude that $x$ and $-x$ both belong to $A = A$. Likewise, if $d(x, B) = d(-x, B) = 0$ then $\pm x \in B$. Finally, if those distances are both positive, then $x$ and $-x$ are neither in $A$ nor in $B$, hence they are both in $C$.

(c) If we don’t assume the sets to be closed: in $S^1$ we can take $A = \{(\cos t, \sin t) | t \in [0, \pi)\}$ and $B = \{(\cos t, \sin t) | t \in [\pi, 2\pi)\}$, which cover $S^1$ and don’t contain antipodal points (in fact the antipodal map exchanges the disjoint half-circles $A$ and $B$). Similarly, in $S^2$, we can take $A = \{(x,y,z) | z > 0 \text{ or } (z = 0 \text{ and } (y > 0 \text{ or } (y = 0 \text{ and } x > 0)))\}$, which is “exactly one half” of the sphere, $B = S^2 - A$ which is the image of $A$ under the antipodal map, and $C = \emptyset$.

With more closed subsets: in $S^1$, take $A = \{(\cos t, \sin t) | t \in [-\pi/3, \pi/3]\}$, $B = \{(\cos t, \sin t) | t \in [\pi/3, \pi]\}$, $C = \{(\cos t, \sin t) | t \in [-\pi, -\pi/3]\}$ (each of them is a closed arc covering one third of the circle; these arcs are centered at the vertices of an equilateral triangle). (Or one could in fact take closed arcs that overlap more – any angle in $[2\pi/3, \pi)$ works). In $S^2$, similarly, take four spherical caps centered at the vertices of a regular tetrahedron, each of which is slightly less than a full hemisphere. One can check that these still cover $S^2$. 