1. (a) Assume $A$ is path-connected and $f : A \to Y$ is continuous. Given any two points $y, y' \in f(A)$, let $a, a' \in A$ be such that $y = f(a)$ and $y' = f(a')$. Since $A$ is path-connected, there exists a path from $a$ to $a'$, i.e. a continuous map $h : [t, t'] \to A$ from an interval to $A$ with $h(t) = a$ and $h(t') = a'$. Then $f \circ h : [t, t'] \to Y$ is continuous (as the composition of two continuous maps), and it is a path in $f(A) \subset Y$ connecting $f \circ h(t) = y$ to $f \circ h(t') = y'$. So $f(A)$ is path-connected – and hence all points of $f(A)$ lie in the same path component of $Y$.

(b) By part (a), if $x, x' \in X$ lie in the same path component $(x \sim x')$ then $f(x), f(x') \in Y$ lie in the same path component $(f(x) \sim f(x'))$. Hence, we can define $\pi_0(f)$ as follows: given $A \in \pi_0(X)$ (an equivalence class, i.e. a path component of $X$), let $x \in X$ be any point in the path component $A$, and define $\pi_0(f)(A) \in \pi_0(Y)$ to be equivalence class of $f(x)$, i.e. the path component of $Y$ which contains $f(x)$. The result of (a) implies that this path component does not depend on the choice of $x \in A$. (This is the usual way in which a map which is compatible with an equivalence relation, i.e. maps equivalence classes to equivalence classes, induces a map on the set of equivalence classes: in other terms, denoting by $[x]$ the equivalence class of $x$, in this case its path component, we have defined $\pi_0(f)$ to map $[x]$ to $[f(x)]$).

(c) We have associated to each topological space $X$ a set $\pi_0(X)$, and to each continuous map $f : X \to Y$ a map $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$. To check that $\pi_0$ is a functor, we need to check that $\pi_0(id_X) = id_{\pi_0(X)}$, and $\pi_0(g \circ f) = \pi_0(g) \circ \pi_0(f)$.

Indeed, the identity map $id_X : X \to X$ maps each point of $X$ to itself, and hence each path component of $X$ to itself, so the induced map $\pi_0(id_X) : \pi_0(X) \to \pi_0(X)$ is the identity map. Moreover, assume $f : X \to Y$ and $g : Y \to Z$ are continuous maps. Then $\pi_0(f)$ maps the path component of $X$ which contains a given point $x$ to the path component of $Y$ which contains $f(x)$, and this in turn gets mapped by $\pi_0(g)$ to the path component of $Z$ which contains $g(f(x))$. Meanwhile, $\pi_0(g \circ f)$ maps the path component of $X$ containing $x$ to the path component of $Z$ which contains $g \circ f(x)$. Thus, we have $\pi_0(g \circ f) = \pi_0(g) \circ \pi_0(f)$, and $\pi_0$ is a functor from the category of topological spaces (with continuous maps) to the category of sets.

2. (51.2) (a) We need to show that all continuous maps $X \to I = [0, 1]$ are homotopic to each other; we do this by showing that every continuous $f : X \to [0, 1]$ is homotopic to the constant map $f_0 : X \to [0, 1]$ defined by $f_0(x) = 0$ for all $x \in X$. This is indeed the case, and an explicit homotopy is given by $F : X \times I \to I$ defined by $F(x, t) = tf(x)$, which is clearly continuous, and satisfies $F(x, 0) = 0 = f_0(x)$ and $F(x, 1) = f(x)$.

(b) Assuming $Y$ is path-connected, we need to show that any two continuous maps from $I = [0, 1]$ to $Y$ are homotopic. First we show that every continuous map $f : I \to Y$ is homotopic to the constant map $I \to Y$ which maps every element of $I$ to $f(0)$. Indeed, consider $F : I \times I \to Y$ given by $F(s, t) = f(st)$, which is continuous. This is a homotopy between the constant map $F(s, 0) = f(0)$ and $F(s, 1) = f(s)$. (In other terms: we have shown that every path in $Y$ can be homotoped (not fixing the end points) to the constant path at its starting point).

Next, given two points $y, y' \in Y$, let $f, f' : I \to Y$ be the constant maps taking the values $f(s) = y$ and $f'(s) = y'$ for $s \in I$. Since $Y$ is path-connected, there exists a path $g : I \to Y$ such that $g(0) = y$ and $g(1) = y'$. We then consider the map $F : I \times I \to Y$ defined by $F(s, t) = g(t)$, which gives a homotopy between $F(s, 0) = g(0) = y = f(s)$ and $F(s, 1) = g(1) = y' = f'(s)$. Thus, any path is
homotopic to a constant path, and any two constant paths are homotopic to each other (again, not fixing the end points); it follows that any two maps \( I \to Y \) are homotopic.

3. (51.3) (a) The identity map of \( I = [0, 1] \) is homotopic to the constant map \( f_0(s) = 0 \) by (the reverse of) the homotopy \( F(s,t) = st \). (\( F(s,0) = 0 = f_0(s) , F(s,1) = s = \text{id}(s) \)). Similarly for the identity map of \( \mathbb{R} \), using exactly the same formula.

(b) Assume \( X \) is contractible, so that \( \text{id}_X \) is homotopic to a constant map \( f_0 : X \to X \) mapping every point \( x \in X \) to the same point \( x_0 \in X \), via a homotopy \( F : X \times I \to X \), i.e. a continuous map such that \( F(x,0) = f_0(x) = x_0 \) and \( F(x,1) = \text{id}_X(x) = x \) for all \( x \in X \). It then follows that every point of \( X \) is in the same path component as \( x_0 \). Indeed the map \( g : I \to X \) defined by \( g(t) = F(x,t) \) is continuous and determines a path from \( g(0) = x_0 \) to \( g(1) = x \). By concatenating paths that connect given points to \( x_0 \), we see that any two points of \( X \) can be joined by a path, i.e. \( X \) is path-connected.

(c) Assume \( Y \) is contractible, and let \( F : Y \times I \to Y \) be a homotopy such that \( F(y,1) = y \) is the identity map and \( F(y,0) = y_0 \in Y \) is a constant map sending every point to some point \( y_0 \in Y \). Then given any map \( g : X \to Y \), we consider \( G : X \times I \to Y \) defined by \( G(x,t) = F(g(x),t) \). This is continuous, and defines a homotopy between \( g \) and the constant map \( g_0 \) which maps every point of \( X \) to \( y_0 \). Indeed, \( G(x,1) = F(g(x),1) = g(x) \) and \( G(x,0) = F(g(x),0) = y_0 \). It follows that every map from \( X \) to \( Y \) is homotopic to the constant map \( g_0 \), and hence that any two maps from \( X \) to \( Y \) are homotopic to each other.

(d) The argument is essentially the same as in part (b) of the previous problem. Assume \( X \) is contractible, i.e. \( \text{id}_X \) is homotopic to a constant map \( g(x) = x_0 \), by a homotopy \( G : X \times I \to X \), \( G(x,0) = x_0 \), \( G(x,1) = x \) for all \( x \in X \).

First we show that every continuous map \( f : X \to Y \) is homotopic to the constant map \( X \to Y \) which maps every element of \( X \) to \( f(x_0) \). Indeed, define a continuous map \( F : X \times I \to Y \) by \( F(x,t) = f(G(x,t)) \). This is a homotopy between the constant map \( F(x,0) = f(G(x,0)) = f(x_0) \) and \( F(x,1) = f(G(x,1)) = f(x) \).

Next, we show that if \( Y \) is path connected then constant maps (sending every point of \( X \) to the same point of \( Y \)) are homotopic to each other. Indeed, given two points \( y_0, y_1 \in Y \), let \( f_0, f_1 : X \to Y \) be the constant maps taking the values \( f_0(x) = y_0 \) and \( f_1(x) = y_1 \) \( \forall x \in X \). Since \( Y \) is path-connected, there exists a path \( g : I \to Y \) such that \( g(0) = y_0 \) and \( g(1) = y_1 \). We then consider the map \( F : X \times I \to Y \) defined by \( F(x,t) = g(t) \), which gives a homotopy between \( F(x,0) = g(0) = y_0 = f_0(x) \) and \( F(s,1) = g(1) = y_1 = f_1(x) \).

Thus, assuming \( X \) contractible and \( Y \) path-connected, any map \( X \to Y \) is homotopic to a constant map, and any two constant maps are homotopic to each other; it follows that any two maps \( X \to Y \) are homotopic to each other.