Math 131 Homework 4 Solutions

1. (23.5) If $X$ has the discrete topology, and $A \subset X$ consists of more than one point, then given
$x \in A$, $\{x\}$ and $A - \{x\}$ are non-empty disjoint subsets of $A$, and open (since every subset is open
in the discrete topology), hence form a separation of $A$: so $A$ is not connected. Thus $X$ is totally
disconnected.

The converse does not hold, and there are many examples of totally disconnected topologies other
than the discrete one. For example $\mathbb{R}_f$: if $x, y \in A$, $x < y$, then $(A \cap (\infty, y)) \cup (A \cap [y, \infty))$
gives a separation of $A$: or $\mathbb{Q}$ as a subspace of $\mathbb{R}$ with the standard topology: if $x, y \in A \subset \mathbb{Q},$
$x < y$, then pick $z \in (x, y)$ irrational, and $(A \cap (\infty, z)) \cup (A \cap (z, \infty))$ gives a separation; or even
$\{0\} \cup \{\frac{1}{n}, n \geq 1\}$ as a subspace of $\mathbb{R}$ with the standard topology (this is not discrete since $\{0\}$ is
not open; the argument given for $\mathbb{Q}$ shows that it is totally disconnected).

2. (23.9) (Strongly recommended: draw a picture to follow alongside the argument!) Observe that
$(X \times Y) - (A \times B) = ((X - A) \times Y) \cup ((X \times (Y - B))$. Thus, fixing $a \in X - A$ and $b \in Y - B$, we
adapt the proof of Theorem 23.6 in Munkres, writing $(X \times Y) - (A \times B)$ as a union of subspaces

$$T_x = (X \times \{b\}) \cup (\{x\} \times Y), \quad \forall x \in X - A,$$

$$T_y = (X \times \{y\}) \cup (\{a\} \times Y), \quad \forall y \in Y - B.$$ 

Since $X \times \{b\}$ and $\{x\} \times Y$ are connected, and have the point $(x, b)$ in common, their union $T_x$ is
connected. Similarly for $T'_y$. Moreover, the subspaces $T_x$ and $T'_y$ all have the point $(a, b)$ in
common, so their union $(\bigcup_{x \in X - A} T_x) \cup (\bigcup_{y \in Y - B} T'_y)$ is connected. Moreover, $\bigcup_{x \in X - A} T_x =
((X - A) \times Y) \cup (X \times \{b\})$ and $\bigcup_{y \in Y - B} T'_y = (\{a\} \times Y) \cup (X \times (Y - B))$, so their union is indeed
$(X \times Y) - (A \times B)$.

3. (24.1) (a) If we remove any point $x \in (0, 1)$ from $(0, 1)$ then $(0, 1) - \{x\} = (0, x) \cup (x, 1)$ is
disconnected; whereas $(0, 1) - \{1\} = (0, 1)$ is connected. Thus $(0, 1)$ and $(0, 1)$ are not homeomorphic
(if there were such a homeomorphism $f : (0, 1) \to (0, 1)$, then the image of $(0, 1) - \{1\}$ would be
$(0, 1) - \{f(1)\}$, but one is connected and the other isn’t). $(0, 1)$ also fails to be homeomorphic to
$[0, 1]$ for the same reason $(0, 1) - \{1\}$ is connected). Finally, $(0, 1)$ and $[0, 1]$ are not homeomorphic
either because removing two points from $(0, 1)$ always produces a disconnected subset, whereas
removing 0 and 1 from $[0, 1]$ gives a connected subset.

(b) $(0, 1)$ embeds into $[0, 1]$ (by the inclusion map $f(x) = x$), while $[0, 1]$ is homeomorphic to say
$[\frac{1}{3}; \frac{2}{3}] \subset (0, 1)$, hence it embeds into $(0, 1)$ (by the map $g(x) = \frac{1}{3} + \frac{x}{3}$. So $(0, 1)$ and $[0, 1]$ embed into
each other but are not homeomorphic.

(c) The complement of a point in $\mathbb{R}$ is disconnected: $\mathbb{R} - \{x\} = (-\infty, x) \cup (x, \infty)$. However, the
complement of a point in $\mathbb{R}^n$ is connected for $n \geq 2$. This follows e.g. from the result of the
previous exercise, writing $\mathbb{R}^n - \{(x_1, \ldots, x_n)\} = (\mathbb{R}^{n-1} \times \mathbb{R}) - (\{(x_1, \ldots, x_{n-1})\} \times \{x_n\})$ (and using
the connectedness of $\mathbb{R}^{n-1}$). Hence $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}$.

4. (24.2) Assume $f : S^1 \to \mathbb{R}$ is continuous. Let $g : S^1 \to \mathbb{R}$ be the map defined by $g(x) = f(x) - f(-x)$, which is also continuous. If $g(x) = 0$ for all $x \in S^1$ then we are done; otherwise, there exists $x \in S^1$ such that $g(x) \neq 0$. Without loss of generality, we can assume that $g(x) > 0$;
and then $g(-x) = f(-x) - f(x) = -g(x) < 0$. Since $S^1$ is connected and $g$ is continuous, the
intermediate value theorem implies the existence of $y \in S^1$ such that $g(y) = 0$, i.e. $f(y) = f(-y)$. 

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Alternative solution: if \( f(x) \neq f(-x) \) for all \( x \in S^1 \), then setting \( U = \{ x \in S^1 \mid f(x) < f(-x) \} \) and \( V = \{ x \in S^1 \mid f(x) > f(-x) \} \), then \( U \) and \( V \) are open (by continuity of \( f \)) and disjoint, and \( S^1 = U \cup V \), so by connectedness one of \( U \) and \( V \) must be all of \( S^1 \) and the other must be empty. This is impossible since \( x \in U \leftrightarrow -x \in V \).

5. (24.3) Let \( f : [0,1] \to [0,1] \) be a continuous function, and consider \( g : [0,1] \to \mathbb{R} \) defined by \( g(x) = f(x) - x \), which is also continuous. \( f \) has a fixed point if there exists \( x \in [0,1] \) such that \( g(x) = 0 \). Observe that \( g(0) = f(0) \geq 0 \) and \( g(1) = f(1) - 1 \leq 0 \); if one of these is zero then we are done; otherwise, \( g(0) > 0 \) and \( g(1) < 0 \) so the intermediate value theorem (i.e. the connectedness of \( g([0,1]) \subset \mathbb{R} \)) implies that there exists \( x \in (0,1) \) such that \( g(x) = 0 \), hence \( f(x) = x \).

On the other hand, there exist continuous maps \( f : [0,1] \to [0,1] \) without a fixed point, for example \( f(x) = \frac{1+x}{2} \). Similarly for \( (0,1) \) (same example).

6. Assume \((X,d)\) is a connected metric space, and let \( x \in X \). For every \( r > 0 \), \( B_r(x) = \{ y \in X \mid d(x,y) < r \} \) is open by definition, but \( U_r(x) = \{ y \in X \mid d(x,y) > r \} \) is also open. Indeed, if \( y \in U_r(x) \) then, letting \( \epsilon = d(x,y) - r > 0 \), we find that \( B_r(y) \subset U_r(x) \). Now, assuming \( X \) has more than one point, let \( y \neq x \), and let \( a = d(x,y) > 0 \). We claim that, for each \( r \in (0,a) \) there exists a point of \( X \) whose distance to \( x \) is equal to \( r \). Indeed, if no point lies at distance \( r \) from \( x \) then \( X = B_r(x) \cup U_r(x) \), where \( B_r(x) \) and \( U_r(x) \) are disjoint (obvious), open (see above), and non-empty \((x \in B_r(x) \text{ and } y \in U_r(x))\), contradicting the connectedness of \( X \). So for all \( r \in (0,a) \) we can find some \( z(r) \in X \) with \( d(x,z(r)) = r \). Since \((0,a) \subset \mathbb{R} \) is uncountable this shows \( X \) is uncountable.

Alternatively: the function \( \delta : X \to \mathbb{R} \) defined by \( \delta(y) = d(x,y) \) is continuous \((d(y_1,y_2) < \epsilon \Rightarrow |\delta(y_1) - \delta(y_2)| < \epsilon \) by the triangle inequality\), so \( \delta(X) \subset \mathbb{R} \) is connected. Picking some \( y \neq x \), let \( a = \delta(y) > 0 \), while \( \delta(x) = 0 \). The intermediate value theorem then implies that \([0,a] \subset \delta(X)\), and this in turn implies that \( X \) is uncountable (else \( \delta(X) \) would be finite or countable).

7. (26.2) (a) Let \( X \subset \mathbb{R} \) with the finite complement topology (which is also the finite complement topology on \( X \)). We assume \( X \neq \emptyset \) (otherwise the statement is obvious). Let \((U_i)_{i \in I}\) be an open cover of \( X \), and pick \( i_0 \in I \) such that \( U_{i_0} \neq \emptyset \). Then \( X - U_{i_0} \) is a finite set: \( X - U_{i_0} = \{ x_1, \ldots, x_n \} \) for some \( n \in \mathbb{N} \) and some \( x_1, \ldots, x_n \in X \). For each \( k = 1, \ldots, n \), \( x_k \in \bigcup U_i \), so there exists \( i_k \in I \) such that \( x_k \in U_{i_k} \). We then find that \( U_{i_0} \cup U_{i_1} \cup \cdots \cup U_{i_n} \) is a finite subcover of the given open cover. This proves that \( X \) is compact.

(b) Consider \([0,1]\) with the countable complement topology, i.e. non-empty open sets are subsets whose complement is finite or countable. For each \( q \in [0,1] \cap \mathbb{Q} \), let \( U_q = ([0,1] - \mathbb{Q}) \cup \{ q \} \) be the subset consisting of all the irrationals and the single rational \( q \). Since rationals are countable, \( U_q \) has countable complement hence is open in this topology. Moreover, the \( U_q \) cover \([0,1]\), since they contain all the irrationals and every rational number \( q \) is in one of these open sets, namely \( U_q \). However, since \( q \) is not in any of the other subsets \( U_{q'} \), \( q' \neq q \), there are no strict subcovers of the collection \( \{U_q\}_{q \in [0,1] \cap \mathbb{Q}} \). Indeed, if we omit some \( U_q \) then the rational \( q \) is no longer in the union of the remaining open sets of the collection. Thus this open cover has no finite subcover, and \([0,1] \) is not compact.

8. (26.7) (Strongly recommended: draw a picture to follow alongside the argument!) Assume \( Y \) is compact, and let \( A \subset X \times Y \) be a closed subset in the product topology. We need to show that \( \pi_1(A) = \{ x \in X \mid \exists y \in Y, (x,y) \in A \} \) is closed in \( X \), i.e. that its complement is open. Let
$x \in X - \pi_1(A)$, i.e. $\exists y \in Y$ such that $(x, y) \in A$. For each $y \in Y$, $(x, y)$ is in the complement of $A$, which is open in $X \times Y$, so there exist neighborhoods $U_y$ of $x$ in $X$ and $V_y$ of $y$ in $Y$ such that $U_y \times V_y$ is disjoint from $A$. The open sets $(V_y)_{y \in Y}$ form an open cover of the compact space $Y$, so there exist $y_1, \ldots, y_n \in Y$ such that $Y = V_{y_1} \cup \cdots \cup V_{y_n}$. Let $U = U_{y_1} \cap \cdots \cap U_{y_n}$, which is a neighborhood of $x$ in $X$. Then $U \times Y = (U \times V_{y_1}) \cup \cdots \cup (U \times V_{y_n}) \subset (U_{y_1} \times V_{y_1}) \cup \cdots \cup (U_{y_n} \times V_{y_n}) \subset (X \times Y) - A$. Thus $A$ contains no points of the form $(x', y')$ for $x' \in U$, hence $U \subset X - \pi_1(A)$. This shows that $X - \pi_1(A)$ is open, hence $\pi_1(A)$ is closed.

9. (26.8) If $f : X \to Y$ is continuous and $Y$ is Hausdorff then the graph $G_f = \{(x, f(x))\} \subset X \times Y$ is closed. Indeed, if $(x, y) \notin G_f$, then $f(x) \neq y$, so there exist disjoint neighborhoods $V \ni f(x)$ and $V' \ni y$ in $Y$. Now, $f$ is continuous so $U = f^{-1}(V) \subset X$ is open, and $x \in U$ since $f(x) \in V$. We now observe that the neighborhood $U \times V'$ of $(x, y)$ is disjoint from $G_f$, since if $(x', y') \in U \times V'$ then $f(x') \in V$ can’t be equal to $y' \in V'$. This implies that the complement of $G_f$ is open, hence $G_f$ is closed.

Conversely, assume $G_f$ is closed, and assume $Y$ is compact. We follow the hint in Munkres. Let $V \subset Y$ be open, and observe that $Y - V$ is closed in $Y$, hence compact. Thus, by the previous exercise, the projection $\pi_1 : X \times (Y - V) \to X$ is a closed map. Applying this to the intersection of $G_f$ with $X \times (Y - V)$, which is closed in the subspace topology, we find that $\pi_1(G_f \cap (X \times (Y - V)))$ is closed in $X$. On the other hand,

$$\pi_1(G_f \cap (X \times (Y - V))) = \{x \in X \mid \exists y \in Y - V \text{ such that } (x, y) \in G_f\}$$

$$= \{x \in X \mid f(x) \in Y - V\} = f^{-1}(Y - V) = X - f^{-1}(V),$$

so we conclude that $f^{-1}(V)$ is open. Hence $f$ is continuous.

The result fails if $Y$ is not compact: for example the map $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 1/x$ for $x \neq 0$ and $f(0) = 0$ is not continuous for the standard topology (e.g. $f^{-1}((-1, 1)) = (-\infty, -1) \cup \{0\} \cup (1, \infty)$ is not open), but its graph $G_f = \{(x, y) \in \mathbb{R}^2 \mid xy = 1\} \cup \{(0, 0)\}$ is closed since it is the union of two closed sets (the hyperbola $\{(x, y) \mid xy = 1\}$ and the single point $\{(0, 0)\}$).

10. $X = \mathbb{R}^n \cup \{\infty\}$ is Hausdorff: given distinct points $x, y \in X$, if $x, y \in \mathbb{R}^n$ then they have disjoint neighborhoods within $\mathbb{R}^n$ (open balls in $\mathbb{R}^n$ of radius less than $|x - y|/2$); whereas if say $y = \infty$ and $x \in \mathbb{R}^n$ has norm $|x| = r$, then the neighborhoods $B_1(x) \subset \mathbb{R}^n$ of $x$ and $U_{r+1} = \{\infty\} \cup \{z \in \mathbb{R}^n \mid |z| > r + 1\}$ of $\infty$ are disjoint (since $z \in B_1(x) \Rightarrow |z| < |x| + 1$).

Let $(V_i)_{i \in I}$ be an open cover of $X$. There is some $i_0 \in I$ such that $\infty \in V_{i_0}$; $V_{i_0}$ thus contains a basis neighborhood of $\infty$ of the form $U_r = \{\infty\} \cup \{x \in \mathbb{R}^n \mid |x| > r\}$ for some $r > 0$. In particular, $X - V_{i_0} \subset A = \{x \in \mathbb{R}^n \mid |x| \leq r\}$, which is closed and bounded in $\mathbb{R}^n$ hence compact (Theorem 27.3). Since the open sets $V_i \cap A$ form an open cover of $A$ and $A$ is compact, there exists a finite subcover, i.e. there exist $i_1, \ldots, i_n \in I$ such that $V_{i_1} \cup \cdots \cup V_{i_n} \supset A$. Then $V_{i_0} \cup V_{i_1} \cup \cdots \cup V_{i_n} \supset V_{i_0} \cup A = X$, i.e. the given open cover of $X$ admits a finite subcover. Hence $X$ is compact.