Math 131 Final Review - list of main topics
(not a complete list!)

1. Topological spaces: $(X, T)$, $T = \{ U \subseteq X \mid U \text{ open} \}$.
   - Basis for a topology on $X$, $B$: a collection of open sets, $\text{basis for } T$.
   - $B(x)$ is the basis for the neighborhood system at $x$.

2. Connectedness & compactness:
   - $X$ is connected if $X = U \cup V$, $U, V$ open disjoint $\Rightarrow$ one is $X$ and the other is $\emptyset$.
   - $f: X \rightarrow Y$ continuous, $X$ connected $\Rightarrow f(X)$ connected.
   - Path-connected: any two points of $X$ can be joined by a path, $f: I \rightarrow X$.
   - $X$ is compact if $\{ U_i \}_{i \in I}$ cover $X$, $\exists$ finite subcover $X = U_i$.
   - $f: X \rightarrow Y$ continuous, $f(X)$ compact $\Rightarrow$ $f(X)$ compact.
   - $X$ compact, $F \subseteq X$ closed $\Rightarrow f(F)$ closed.
   - $X$ Hausdorff, $X$ compact $\Rightarrow X$ compact.

3. Exam:
   - OH today 12:30-2, tomorrow 9-10:30.
   - Next Monday 12:30-2.
   - Exam on Tue 12/17, 9-12, Norbeck B 101.
   - OK: Pencils, may use or Beckham's notes.
   - Course evals!
(finite) products of \{compact\} spaces are \{compact\}.

If \((X,d)\) metric space & compact then:

- every open cover \(X = \bigcup U_i\) has a Lebesgue number \(\delta > 0\); \(\text{diam}(A) < \delta \Rightarrow \exists i \text{ s.t. } A \subseteq U_i\).
- every continuous function \(f:(X,d) \rightarrow (Y,d_y)\) is uniformly continuous.

For metric spaces, compact \(\iff\) limit point compact \(\iff\) sequentially compact.

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\begin{align*}
\text{compact} & \iff \text{limit point compact} & \iff \text{sequentially compact} \\
& \iff (\text{every} \text{ infinite subset has a limit point}) & & (\text{every sequence has a convergent subsequence}) \\
& (\text{always } \Rightarrow) & & (\text{always } \Leftarrow).
\end{align*}
\]

A one-point compactification of \(X\) is a compact space \(Y\) s.t. \(Y - \{\infty\} \cong X\).

Build:\( Y = X \cup \{\infty\}\), open = that of \(X\)

+ complements of compact subsets of \(X\).

If \(X\) is locally compact \((\forall x \in X \exists \text{bdry } U \ni x\) and compact \(C \supseteq U \ni x\) and Hausdorff

then \(Y\) is Hausdorff and unique up to homeo.

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Homotopy and Fundamental Group:

- categories, functors (language only).
- homotopy: \(f_0, f_1: X \rightarrow Y\) continuous; a homotopy is \(F: I \times X \rightarrow Y\) continuous, \(F|_{X \times 0} = f_0, F|_{X \times 1} = f_1\).

- paths \(f_0, f_1: I \rightarrow Y\) are path-homotopic if \(\exists\) homotopy \(F: I \times I \rightarrow Y\) fixing end points.

- path-homotopy classes of paths \(Y\) in \(X\) form a groupoid for path composition \(f \circ g\)

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\begin{align*}
& \begin{array}{ccc}
Y & \overset{f}{\longrightarrow} & Y \\
\downarrow & & \downarrow \phi \\
\downarrow & & \downarrow \phi \\
Z & \overset{g}{\longrightarrow} & Z
\end{array} \\
\end{align*}
\]

- loops based at \(x_0\) \((=\text{paths } x_0 \rightarrow x_0)\) \(=\) fundamental group \(\pi_1(X, x_0)\)

(pullback composition, identity = constant loop, inverse = reverse loop)

- \(x_0, x_1 \in \text{same path component of } X \Rightarrow \pi_1(X, x_0) = \pi_1(X, x_1)\) (by attaching path \(\alpha \ast f \ast \alpha^{-1}\)).

- \(f: (X, x_0) \rightarrow (Y, y_0)\) induces homomorphism \(f_\# : \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0)\) fundamental \((f_\#)|_x = f_\circ g_x\).

- Ex: \(\mathbb{R}^n\), convex subsets of \(\mathbb{R}^n\), \(n \geq 2\) are simply connected \((\pi_1 = \{1\})\).

\[
\begin{align*}
\pi_1(S^1, b_0) & \cong \mathbb{Z}, \pi_1(\mathbb{C}, b) \cong \mathbb{Z}^2, \pi_1(\mathbb{C}, \infty) \cong \text{free group } \langle a, b \rangle
\end{align*}
\]

- Applications of \(\pi_1(S^1) \cong \mathbb{Z}\):

  - \# retraction \(r: B \rightarrow S^1\) (ie. \(r\) continuous, \(r|_{S^1} = \text{id}_{S^1}\)).

  - every continuous \(f: B^2 \rightarrow B^2\) has a fixed pt \((f(x) = x)\) (Brouwer).

  - \(f: S^2 \rightarrow \mathbb{R}^2\) continuous \(\Rightarrow\) \(\exists x \text{ s.t. } \forall x \neq f(x) = f(-x)\) (Borsuk-Ulam).

- Deformation retraction: \(r: X \rightarrow A\) retraction, \((roi = id_A)\) s.t. \(ioi\) is homotopic to \(id_X\)

  among maps that leave \(A\) fixed, \(\text{i.e. } H: X \times I \rightarrow X, H(x, 0) = x\)

  Then \(\pi_1(A, x_0) = \pi_1(X, x_0)\) \((\text{i.e., inverse images})\)

  \(H(x, 1) \in A, \forall x \in X\)

  \(H(a, 0) = a, \forall a \in A\)

- The same holds more generally for homotopy equivalences \(f: X \rightarrow Y, g \circ f \cong id_X, f \circ g \cong id_Y\).
• Covering spaces: \( p: E \to B \), \( \forall b \in B \exists U \ni b \) evenly covered by \( p \) 

\[ (p^{-1}(U)) = \text{disjoint union of slices } V_x, \text{ each } p|_{V_x} : V_x \to U. \]

• Every path \( f: I \to B \) starting at \( b_0 \) has unique lift \( \tilde{f}: I \to E \) starting at \( e_0 \in p^{-1}(b_0) \). (Path) homotopy lifts to (path) homotopy.

• Looking at end points of lifts of loops in \( (B, b_0) \), get lifting map \( \pi_1(B, b_0) \to p^{-1}(b_0) \).

• Those loops which lift to a loop in \( (E, e_0) \) form a subgroup \( H \subset \pi_1(B, b_0) \), and 

\[ \pi_1(E, e_0) \cong H. \]

• A map \( g: (Y, y_0) \to (B, b_0) \) lifts to \( \tilde{g}: (Y, y_0) \to (E, e_0) \) iff \( \tilde{g}(\pi_1(Y, y_0)) \subset H \).

• Classification of covering spaces (up to equivalence) is class of subgroups \( H \subset \pi_1(B) \) (up to conjugacy).

• Universal cover: simply-connected \( E \) (i.e., \( \pi_1 = \{1\} \)).

• Van Kampen: \( X = U \cup V \), \( U, V \) open, \( U \cap V \neq \emptyset \) path-connected \( \Rightarrow \)

\[ \pi_1(X, x_0) \text{ is generated by the images of } i_1: \pi_1(U) \to \pi_1(X), \ i_2: \pi_1(V) \to \pi_1(X) \]

• if \( \pi_1(U \cap V) = \{1\} \) then \( \pi_1(X) \) is the free product \( \pi_1(U) \ast \pi_1(V) \)

• Otherwise, quotient by smallest normal subgroup that makes \( i_1(g) = i_2(g) \) \( \forall g \in \pi_1(U \cap V) \).