What is topology? Unlike geometry, which concerns quantitative information about space (distance, volume, ...), topology concerns itself with qualitative properties that are invariant under continuous deformation.

Eg: is it connected? (a single piece)
simple connected?

\[ \bigcirc \ \text{vs.} \ \bigcirc \]

punched torus \[ \cong \ \bigcirc \bigcirc \bigcirc \]

Eg: how/why is a Möbius band different from a regular band?

\[ \bigcirc \ \text{vs.} \ \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \]

[orientability? boundary?]

Algebraic topology associates invariants to topological spaces that help tell them apart.

We'll get a taste of it in the 2nd part of the course, focusing on the fundamental group.

But first need the language of point-set topology:

- topological spaces, open & closed sets,
- compactness, connectedness.
Topological spaces are sets equipped with data that lets us talk about continuity — notion of “nearness” so we can talk about limits etc.

Example: extreme value theorem says: $f: [a,b] \to \mathbb{R}$ continuous $\Rightarrow f$ achieves its max and min at some points of $[a,b]$.

This is in fact true for any continuous $f: X \to \mathbb{R}$ whenever $X$ is a compact topological space, and is a special instance of

**Theorem:** If $f: X \to Y$ continuous mapping between topological spaces, & $X$ compact, then $f(X)$ is compact.

Since the general notion of topological space is quite abstract, let’s start with a more familiar class of examples: **METRIC SPACES**

**Def:** A metric space $(X,d)$ is a set $X$ together with a distance function $d: X \times X \to \mathbb{R}_{\geq 0}$ such that:

1) For $p, q \in X$, $d(p, q) = 0 \iff p = q$

2) $d(p, q) = d(q, p)$

3) For $p, q, r \in X$, $d(p, r) \leq d(p, q) + d(q, r)$ (triangle inequality)

Example: $X = \mathbb{R}^n$ with Euclidean distance $d(x, y) = \left( \sum_{i=1}^{n} (y_i - x_i)^2 \right)^{1/2}$.

If $Y \subseteq \mathbb{R}^n$ then $(Y, d|_Y)$ is a metric space. (“induced metric”).

Example: different metrics on $\mathbb{R}^n$:

- $d_1(x, y) = \sum_{i=1}^{n} |y_i - x_i|$

- $d_\infty(x, y) = \max\{ |y_i - x_i| \}$

Exercise: show $(\mathbb{R}^n, d_1)$ & $(\mathbb{R}^n, d_\infty)$ are metric spaces.

**Open sets:**

**Def:** $(X, d)$ metric space, $p \in X$, $r > 0$: the open ball of radius $r$ around $p$ is the open ball of radius $r$ around $p$ is the open ball $\mathcal{B}_r(p) = \{ q \in X \mid d(p, q) < r \}$ (or neighborhood).

- $U \subseteq X$ is open if $\forall p \in U$, $\exists r > 0$ st: $\mathcal{B}_r(p) \subseteq U$.

Facts: open balls are open, so are arbitrary unions & finite intersections of open sets. (Homework)
**Closed sets & limits:**

**Def:** A sequence \( p_1, p_2, \ldots \) in \( X \) converges to a limit \( p \in X \) (write \( p_n \to p \) or \( \lim_{n \to \infty} p_n = p \)) if \( \forall \varepsilon > 0 \ \exists N \ \text{st.} \ \forall n \geq N, \ d(p_n, p) < \varepsilon \).

(unique if it exists).

**Def:** A sequence \( p_1, p_2, \ldots \) in \( X \) is Cauchy if \( \forall \varepsilon > 0 \ \exists N \ \text{st.} \ \forall m, n \geq N \ d(p_m, p_n) < \varepsilon \).

Exercise: if a sequence converges then it is Cauchy, but not necessarily vice-versa.

A metric space is complete if every Cauchy sequence converges.

**Ex.** \( \mathbb{R} \) is complete, but \( \mathbb{Q} \) (with induced metric) isn't complete.

**Def:** \( Z \subseteq X \) is closed if its complement \( X \setminus Z \) is open.

\( \triangleright \) Most subsets of \( X \) are neither open nor closed !!

\( \emptyset \) and \( X \) are both open and closed !

**Prop.** \( Z \subseteq X \) is closed if and only if:

\( \forall \) sequence \( \{p_n\} \) in \( Z \) which converges to a limit \( p \in X \), then \( p \in Z \).

\( \triangleright \) true in all topological spaces, but \( \Leftarrow \) only holds in sufficiently nice one (such as metric spaces).

**Proof:** if \( Z \) is not closed then \( X \setminus Z \) not open, ie. \( \exists p \in X \setminus Z \) st. \( \forall r > 0 \ , \ B_r(p) \not\subseteq X \setminus Z \)

For this point \( p \), \( \forall n \geq 1 \ , \ \exists p_n \in Z \) with \( d(p_n, p) < \frac{1}{n} \).

This gives a sequence \( p_n \to p \) in \( Z \), \( p \notin Z \).

* Conversely, if \( \exists p_n \in Z \) \( p \in X \setminus Z \) \( p_n \to p \), then

\( \forall r > 0 \ \exists N \ \text{st.} \ \forall n \geq N \ d(p_n, p) < r \).

So \( N_r(p) \) contains points of \( Z \), hence \( N_r(p) \not\subseteq X \setminus Z \).

Hence \( X \setminus Z \) isn't open, ie. \( Z \) isn't closed.

**Continuity:** **Def.** \( (X, d_X) \), \( (Y, d_Y) \) metric spaces. \( f : X \to Y \) is continuous if

\( \forall p \in X \ , \ \forall \varepsilon > 0 \ , \ \exists S > 0 \ \text{st.} \ d_X(p, x) < S \Rightarrow d_Y(f(p), f(x)) < \varepsilon \).
**Theorem:** \( f: X \to Y \) is continuous iff \( \forall U \subseteq Y \) open, \( f^{-1}(U) \subseteq X \) is open.

**Proof:** Assume \( f \) continuous, let \( U \subseteq Y \) open, let \( p \in f^{-1}(U) \), i.e. \( f(p) \in U \).

Want: \( \exists \delta > 0 \) s.t. \( B_\delta(p) \subseteq f^{-1}(U) \).

Known: \( \exists \varepsilon > 0 \) s.t. \( B_\varepsilon(f(p)) \subseteq U \). (since \( U \) open).

By continuity, \( \exists \delta > 0 \) s.t. \( d(p, x) < \delta \Rightarrow f(x) \in B_\varepsilon(f(p)) \subseteq U \).

Hence \( B_\delta(p) \subseteq f^{-1}(U) \). So \( f^{-1}(U) \) is open.

Conversely, assume \( U \subseteq Y \) open \( \Rightarrow \exists \delta > 0 \). Fix \( p \in X \), \( \exists \varepsilon > 0 \). \( B_\varepsilon(f(p)) \) is open in \( Y \), so \( f^{-1}(B_\varepsilon(f(p))) \) open in \( X \). Hence \( \exists \delta > 0 \) s.t. \( B_\delta(p) \subseteq f^{-1}(B_\varepsilon(f(p))) \).

This means \( d(p, x) < \delta \Rightarrow x \in f^{-1}(B_\varepsilon(f(p))) \Rightarrow f(x) \in B_\varepsilon(f(p)) \). \( \checkmark \)

* Our goal will be to reformulate / generalize all this in the context of **topological spaces**, i.e. sets equipped with a **topology** which may or may not come from a metric.

**Def.** A topology \( T \) on a set \( X \) = collection of subsets of \( X \), which we'll declare to be the **open sets** in \( X \). Needs to satisfy axioms:

- \( \emptyset \in T \), \( X \in T \)
- any union of elements of \( T \) is in \( T \)
- the intersection of finitely many elements of \( T \) is in \( T \).

**Why bother?** One answer: many natural topologies do not come from a metric!

E.g., in analysis:

- on space of (bounded) functions \( f: X \to \mathbb{R} \),
  - uniform convergence topology come from a metric (\( d(f, g) = \sup_{x \in X} |f(x) - g(x)| \))
  - but pointwise convergence \( (f_n \to f) \) iff \( \forall x \in X \) \( f_n(x) \to f(x) \) doesn't. ("product topology")
- \( C^0 \) topology on smooth functions \( \mathbb{R} \to \mathbb{R} \) doesn't come from a metric either.
A topology \( T \) on a set \( X \) is a collection of subsets of \( X \), which we call open sets in \( X \). It needs to satisfy axioms:

1. \( \emptyset \in T \), \( X \in T \)
2. any union of elements of \( T \) is in \( T \)
3. the intersection of finitely many elements of \( T \) is in \( T \).

A space equipped with a topology is called a topological space.

As in metric space, we define \( F \subseteq X \) to be closed if \( F^c = X \setminus F \) is open.

Easy to check: (1) \( \emptyset, X \) are closed.
2. arbitrary intersections of closed sets are closed
3. finite unions are closed

Example: \((X,d)\) metric space, the collection of all open subsets in the sense of lecture 1:
\[ T = \{ U \subseteq X \mid \forall x \in U \exists r > 0 \text{ s.t. } B_r(x) \subseteq U \} \]

is a topology - satisfies the 3 axioms.

Thus, metric spaces are topological spaces.

(Ex: "standard topology" on \( \mathbb{R} \)).

Ex.: if \( X = \{ a, b \} \), then \( T \) must contain \( \emptyset \) and \( \{ a, b \} \)
\( \emptyset \) may or may not contain \( \{ a \} \), or \( \{ b \} \).

\( X = \{ a, b, c \} \): many choices eg. \( T = \{ \emptyset, X \} \), \( T = \{ \emptyset, \{ a \}, X \} \), \( T = P(X) \).

But must contain unions & intersections of its elements eg. if \( \{ a \}, \{ b \} \in T \) then \( \{ a, b \} \in T \).

Ex. if \( X \) is any set, \( P(X) = \{ \text{all subsets of } X \} \) is a topology - called the discrete topology - every set is open.

Ex. Let \( T = \{ S \subseteq X \mid X \setminus S \text{ is finite, or } S = \emptyset \} \).

This is a topology: 1) \( \emptyset \in T \), \( X \in T \) since \( X \setminus X = \emptyset \) finite
("finite complement" or "cofinite" topology).

2) if \( S = \bigcup S_i \), \( S_i \in T \), then \( X \setminus S = \bigcap (X \setminus S_i) \) is finite (or = \( X \)) so \( S \in T \).

3) similarly for finite \( \bigcap \).

Ex. \( X \) infinite set, \( T = \{ S \subseteq X \mid S \text{ finite or } S = X \} \) is not a topology.

Indeed, \( Y \subseteq X \) infinite \( \Rightarrow Y = \bigcup_{y \in Y} \{ y \} \) union of elements of \( T \), but \( Y \notin T \).
Keeping track of all the open sets is cumbersome—in metric space we started with open balls & got a characterization of open sets in terms of these.

The analogous notion for a general topology is that of basis.

**Def:** A basis $B \subseteq P(X)$ is a collection of subsets of $X$ such that:

1. $\bigcup_{B \in B} B = X$
2. if $B_1, B_2 \in B$ and $x \in B_1 \cap B_2$ then $\exists B \in B$ st. $x \in B \subseteq B_1 \cap B_2$.

$B$ is usually not a topology itself, but we can use it to construct one:

**Def:** The topology $T$ generated by a basis $B$ is defined by

$U \in T \iff \forall x \in U \exists B \in B \text{ st. } x \in B \subseteq U$.

**Prop:** $T$ is indeed a topology.

*Proof:* clearly $\emptyset \in T$. $x \in T$ because $\forall x \in X \exists B \in B \text{ st. } x \in B$ (1st axiom).

- if $U_i \in T$ and $x \in U \subseteq U_i$, then $\exists B_i \in B \text{ st. } x \in B_i$, so $\exists B \in B \subseteq U_i \subseteq U_i$.
- $U \in T$.
- if $U_1, U_2 \in T$ and $x \in U_1 \cap U_2$, then $\exists B_1, B_2 \in B \text{ st. } x \in B_1 \subseteq U_2$, $2^{nd}$ axiom $\Rightarrow \exists B \in B \text{ st. } x \in B \subseteq B_1 \cap B_2 \subseteq U_1 \cap U_2$. So $U_1 \cap U_2 \in T$ (finite intersections = repeat this).

**Ex:** $B = \{B_r(x) \mid x \in \mathbb{R}^n, r > 0\}$ (open balls) is a basis, and generates the standard topology of $\mathbb{R}^n$ [this also works in any metric space!]

The set of all “rectangles” $(a_1, b_1) \times (a_2, b_2) \times \ldots \times (a_n, b_n)$ ($a_i < b_i \in \mathbb{R}$) is also a basis for the standard topology of $\mathbb{R}^n$.

(or could take $B =$ all open subsets of $\mathbb{R}^n$, this is also a basis! though not a very useful one... ) (more useful bases have fewer, simpler subsets).

**Prop:** the topology generated by $B$ is $T = \{\text{ all unions of elements of } B\}$ (including empty union = $\emptyset$).
pf: if \( U \in T \) then \( \forall x \in U \ \exists B \in B \cup \{x\} \ s.t. \ x \in B \cup \{x\} \).

Then \( U = \bigcup_{B \in B} B \). Indeed, this is clearly \( U \), and every pt of \( U \) is in some \( B \cup \{x\} \).

so \( U \) is a union of elements of \( B \).

\[ \{ \text{conversely: all elements of } B \text{ are in } T \ldots \text{and hence so are unions thereof.} \} \]

(Indeed: \( B \in B \Rightarrow \forall x \in B, \ x \in B \cup \{x\} \))

Remark: \( T \) is therefore the smallest collection of subsets of \( X \) which is a topology and contains \( B \).

Note: basis for a given topology is not unique.

A expansion of a given open set as union of elements of the basis is not unique.

---

**Example:** \( R \) with usual topology: basis \( = \{ \text{balls } B_r(x) \} = \{ \text{open intervals } (a,b) \} \).

Every open set in \( R \) is a union of open intervals.

If \( U \subset R \) is open, we may need infinitely many intervals to describe it.

\( U = R \setminus \mathbb{Z} = \bigcup_{n \in \mathbb{Z}} (n, n+1) \ldots \)

\[ U = \bigcup_{n \in \mathbb{Z}} \left( \frac{1}{n+1} \right) ^{2n} \left( \frac{1}{n+1/n+1} \right) \ldots \]

Eg: complement \( C \) Cantor set in \( (0,1) = \bigcup_{n \in \mathbb{Z}} (0,1) \) with a 1 in their base 3 expansion:

\( U_1 = \left( \frac{1}{3}, \frac{2}{3} \right) \)

\( U_2 = \left( \frac{1}{3}, \frac{2}{3} \right) \cup \left( \frac{7}{9}, \frac{8}{9} \right) \)

\( \ldots \) (middle thirds of all remaining gaps at each step)

Q: are any open sets in \( R \) the disjoint union of uncountably many open intervals?

Ex: A different topology on \( R \): "lower limit topology" \( T_l \) gen. by basis \( \{ (a,b) | a < b \} \).

\( [a,b] \) is not open in the standard topology \( T \), while it is in \( T_l \). So \( T_l \neq T \).

However, \( (a,b) \) is open in \( T_l \) -- indeed, \( \forall x \in (a,b), \ x \in \{ x, b \} \subset (a,b) \).

\( (a,b) = \bigcup_{x \in (a,b)} \{ x, b \} \)

So \( T \subset T_l \), say \( T_l \) is finer than \( T \). \( T \) is coarser.

(Will see: harder for a sequence to converge to a limit in \( T \).)

Match open: fewer open
Lemmas: Let $B, B'$ be bases for $T, T'$ on $X$. Then $T'$ is finer than $T$ (i.e. $T \subset T'$) if $B \subset T'$ and $B' \subset T'$.

If $T \subset T'$ then $B \subset T \subset T'$.

If $B \subset T'$ then $T = \{ \bigcup B_i : B_i \in B \} \subset T'$, where $B_i$ are open in $T$.

This is the definition of $B$ being open in the topology generated by $B'$.

Example: "French train distance" on $\mathbb{R}^2$:

$$d_T(p, q) = \begin{cases} d(p, q) & \text{if } p, q \text{ on same ray from origin} \\ d(p, 0) + d(0, q) & \text{otherwise} \end{cases}$$

Fact: this is a metric!

What do open balls look like?

E.g. around $(1, 0)$:

- Balls in $d_T$ are (usually) not open for the standard topology, so $T \subset \not T$.
- However, $d_T(p, q) \geq d(p, q)$ for all $p, q \in \mathbb{R}^2$, so $B_{d_T}(p) \subset B_d(p)$.

Hence: if $U$ is open in standard topology, then $\forall x \in U \exists r > 0$ such that $B_r(x) \subset U$, and then $x \in \bigcap B_{d_T}(x) \subset \bigcap B_d(x) \subset U$.

So $U$ is open for $d_T$.

Hence: $T \subset T'$, the "French train topology" is finer than the standard one.
Recall: a basis \( B \) generates a topology \( \mathcal{T} = \{ \bigcup_{x \in U} \exists B \in B / x \in B \subseteq U \big\} \)

\( \{ \bigcup B_i, B_i \in \mathcal{B} \} \).

(\( \mathcal{T} = \text{coarsest topology for which all sets of } \mathcal{B} \text{ are open} \). (eg balls in a metric space)

---

**Example:** the **subspace topology** (Munkres §16)

- X topological space, \( A \subset X \).
- If \( X \) metric space then \( A \) inherits a metric by restriction and then \( B^A_r(p) = B^X_r(p) \cap A \)
  \( V \subset A, \forall r > 0 \)

and, taking unions of balls, open subsets of \( A \) are (open in \( X \)) \( \cap \) A.

This motivates the following definition when \( X \) is a topological space:

**Def:** the **subspace topology** on \( A \) is defined by \( \mathcal{T}_A = \{ U \cap A / U \in \mathcal{T}_X \} \).

Easy to check:
1. This is a topology on \( A \).
2. If \( B \) is a basis for \( \mathcal{T}_X \), then \( B_A = \{ B \cap A / B \in \mathcal{B} \} \) is a basis for \( \mathcal{T}_A \).

**Ex:** in \( [0,1] \subset \mathbb{R} \) with subspace topology, \( (-\frac{1}{2}, \frac{1}{2}) \cap [0,1] = [0, \frac{1}{2}) \) is open.

- Subspace topology for \( \mathbb{R} = \{ x \text{-axis} \} \subset \mathbb{R}^2 \), standard top. is standard topology.

---

**Example:** the **product topology** on \( X \times Y \), given topologies \( \mathcal{T} \) on \( X \) and \( \mathcal{T}' \) on \( Y \):

\( \mathcal{T}' \) is generated by the basis \( \mathcal{B} = \{ U \times V / U \in \mathcal{T}, V \in \mathcal{T'} \} \).

**Claim:** \( \mathcal{B} \) is a basis.

**Proof:**
1) \( X \times Y \in \mathcal{B} \), so the union covers \( X \times Y \)
2) \( (U \times V_1, U_2 \times V_2) \in \mathcal{B} \), then \( (U \times V_1) \cap (U_2 \times V_2) = (U \cap U_2) \times (V_1 \cap V_2) \in \mathcal{B} \)

(recall: would be enough: intersection of two elements of \( \mathcal{B} \) is a union of elements of \( \mathcal{B} \)).

**Remark:** not every open subset in the product topology is of the form \( U \times V \), \( U \in \mathcal{T}, V \in \mathcal{T}' \)), eg. look at \( (U_1 \times V_1) \cup (U_2 \times V_2) \) in above example!

Note: in fact a slightly more efficient basis for the product topology.
Claim: if \( B, B' \) are bases for \( T, T' \) respectively, then \( D = \{ B \times B' / B \in B, B' \in B' \} \) is a basis for the product topology.

**Pf.** \( X \subseteq \{ U \times V / U \in T, V \in T' \} \), but \( U \in T \Rightarrow U = \bigcup_{i \in J} B_i, B_i \in B \) then \( U \times V = \bigcup_{i \in J, j \in J} B_i \times B_j \) so \( D \) generates the same topology.

**Ex.** \( \mathbb{R} \times \mathbb{R} \), the product topology coincides with the standard topology of \( \mathbb{R}^2 \).

Indeed: a basis for product topology consists of rectangles \( (a_1, b_1) \times (a_2, b_2) \) which are all open in the standard topology.

- Conversely, check that the elements of a basis for the standard topology are open in the product topology. This can be done for balls of the Euclidean metric:
  
  
  ![Euclidean Ball Diagram](image)

  But even more easily for the balls of the metric \( d(0) \), which defines the same topology as the Euclidean one (see HW1): balls for \( d(0) \) are rectangles \( (x-r, x+r) \times (y-r, y+r) \) so open in the product topology.

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**Example (Number 514):** the order topology

Let \( X \) be a set with a total ordering, i.e.:

1. \( \forall a, b \in X, \) exactly one of \( a < b, \ b < a, \ a = b \) holds
2. \( \forall a, b, c \in X, \) \( a < b \) and \( b < c \Rightarrow a < c \).

The order topology on \( X \) is generated by the basis consisting of open intervals \( (a, b) = \{ x \in X / a < x < b \} \), and possibly half-open intervals if \( X \) has a smallest element \( a_0 \), [\( a_0, b \)]: a largest element \( b_0 \), (\( a, b_0 \]).

On \( \mathbb{R} \) (with usual order) this generates the standard topology.

**Ex:** let \( I = [0, 1] \), equip \( I \times I \) with the dictionary order: \( a < b \) iff \( a < a' \) or \( a = a' \) and \( b < b' \).

Then basis elements look like:

![Dictionary Order Diagram](image)
Open sets in the standard topology are also not always open in order topology!

eg. Ball for Euclidean metric:

ball centered at \((\frac{1}{2}, 0)\) is not open in the order topology, because any open interval containing \((\frac{1}{2} \cdot 0)\)

must be \((a \cdot b, a' \cdot b')\) with \(a \cdot b < \frac{1}{2} \cdot 0 \Rightarrow a < \frac{1}{2}\) and \(a' \geq \frac{1}{2}\)

so contains a whole vertical interval \(x \cdot \mathbb{I}\) for \(a < x < a'\).

So this topology is not comparable to the standard one (neither is contained in the other).

Continuous functions (Ndaka beginning of §18)

**Def:** A function \(f: X \to Y\) between topological spaces is continuous if

\(\forall U \subseteq Y\) open, \(f^{-1}(U) \subseteq X\) is open.

We've seen that, for metric spaces, this agrees with the \(\varepsilon\)-\(\delta\) definition of continuity.

\(\forall x \in X, \forall \varepsilon > 0 \exists \delta > 0\) s.t. \(B_\delta(x) \subseteq f^{-1}(B_\varepsilon(f(x)))\). (cf. lecture 5).

**Ex:** \(f: \mathbb{R} \to \mathbb{R}\) defined by \(f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}\)

is discontinuous (for the standard topology). What goes wrong with \(f^{-1}(\text{open sets})\)?

**Ans:** eg. \(f^{-1}((0, 1]) = [0, \infty)\) not open.

On the other hand, \(f\) is continuous if we equip \(\mathbb{R}\) with the lower limit topology!

Indeed, \(f^{-1}\) of any subset of \(\mathbb{R}\) is one of \(\emptyset, (-\infty, 0), [0, \infty), \mathbb{R}\) - all of which are open in the lower limit top’. So \(f: \mathbb{R}_l \to \mathbb{R}_l\) is continuous.

**Ex:** \(f: \mathbb{R} \to \mathbb{R}_l\) identity function \(f(x) = x\).

wrt top lower limit top open in \(\mathbb{R}_l\)

This isn't continuous, since \(f^{-1}([0, 1]) = (0, 1)\) isn't open in the standard topology.

However, the identity function \(\mathbb{R}_l \to \mathbb{R}\) is continuous, since \(\mathbb{R}_l\) has a finer topology:

\((U \subset \mathbb{R} \text{ open } \Rightarrow f^{-1}(U) = U \text{ open in } \mathbb{R}_l)\).

**Ex:** \(X, Y\) top spaces, equip \(X \times Y\) with the product topology.

let \(p_2: X \times Y \to X\) first projection. Then \(\forall U \subseteq X\) open, \(p_2^{-1}(U) = U \times Y\)

is open in \(X \times Y\), hence \(p_2\) is continuous.
**Def:** A function \( f: X \to Y \) between topological spaces is **continuous** if
\[ \forall U \subset Y \text{ open}, \quad f^{-1}(U) \subset X \text{ is open}. \]

(Various examples seen last time)

* It suffices to check continuity on elements of a basis!

**Prop:** \( f: X \to Y \) is continuous iff \( f^{-1}(B) \subset X \) is open for all \( B \) in a basis for the topology on \( Y \).

**Proof:** (i) \( f^{-1}(B) \) open \( \forall B \in \text{basis} \) is obviously necessary for continuity of \( f \), since every basis element is open in \( Y \).

- Every open \( U \subset Y \) can be written as \( U = \bigcup_{i \in I} B_i \), \( B_i \in \text{basis} \). Since \( f^{-1}(U) = \bigcup_{i \in I} f^{-1}(B_i) \), if \( f^{-1}(B_i) \) are all open in \( X \) then so is \( f^{-1}(U) \).

Ex: \( X \) any top. space, \( Y \) metric space, then to check continuity of \( f: X \to Y \) it is enough to check that \( f^{-1}(B_r(y)) \) is open \( \forall y \in Y \forall r > 0 \).

---

**Properties of continuous functions:** For any topological spaces:

**Thm:**
1) constant functions \( f: X \to Y \), \( f(x) = y_0 \ \forall x \in X \) for some fixed \( y_0 \in Y \) are continuous.

2) if \( A \subset X \) is given the subspace topology, then the inclusion \( i: A \to X \) is continuous.

3) if \( f: X \to Y \) and \( g: Y \to Z \) are continuous, then \( gof: X \to Z \) is continuous.

4) if \( X = \bigcup_{\alpha} U_{\alpha} \) with \( U_{\alpha} \) open ("open cover of \( X \)"), and \( f: X \to Y \) a function such that \( f|_{U_{\alpha}}: U_{\alpha} \to Y \) is continuous for all \( \alpha \), then \( f \) is continuous with subspace topology.

**PF:**
1) if \( f \) is a constant function then \( \forall U \subset Y \), \( f^{-1}(U) = \{ y_0 \} \) always open.

2) \( \forall U \subset X \text{ open} \), \( f^{-1}(U) = U \cap A \) is open in \( A \).

3) \( \forall U \subset Z \), \( (gof)^{-1}(U) = f^{-1}(g^{-1}(U)) \) \( \forall f(x) \in U \iff f(x) \in g^{-1}(U) \iff x \in f^{-1}(g^{-1}(U)) \) \( U \text{ open in } Z \Rightarrow g^{-1}(U) \text{ open in } Y \Rightarrow f^{-1}(g^{-1}(U)) \text{ open in } X \).
4) \( V \subset X \) open, \( (f|_{U_\alpha})^{-1}(V) = f^{-1}(V) \cap U_\alpha \), so \( f^{-1}(V) = \bigcup_\alpha (f|_{U_\alpha})^{-1}(V) \). 

\( (f|_{U_\alpha})^{-1}(V) \) is open in \( U_\alpha \), so it's the intersection of \( U_\alpha \) with an open subset of \( X \), hence (since \( U_\alpha \) also open) it's an open subset of \( X \).

\( f^{-1}(V) \) is therefore a union of open sets in \( X \), hence open. 

\( \square \)

---

**Homeomorphisms**: two topological spaces \( X \) and \( Y \) are "homeomorphic" if they are topologically the same — namely, if there exists a bijection \( f: X \to Y \) s.t. \( U \) open \( \iff f(U) \) is open.

**Def.** A bijection \( f: X \to Y \) is a **homeomorphism** if \( f \) and \( f^{-1} \) are both continuous.

Say \( X \) and \( Y \) are **homeomorphic** if there exists a homeomorphism between them.

**Ex.** we've seen that a continuous bijection need not be a homeomorphism.

e.g. \( f=\text{id}: \mathbb{R}_{\infty} \to \mathbb{R} \) is continuous \((\alpha, \beta) \subset \mathbb{R}_{\infty} \) open \( \iff f((\alpha, \beta)) \subset \mathbb{R} \) open \( \), but \( f^{-1} \) isn't \((\alpha, \beta) \subset \mathbb{R}_{\infty} \) not open \( \).

**Ex.** \( X = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N} \} \) with topology induced by metric of \( \mathbb{R} \)

\( N = \{0, 1, 2, 3, \ldots\} \) with discrete topology

Define \( f: N \to X \) by \( f(0) = 0 \), \( f(n) = \frac{1}{n} \) for \( n \geq 1 \).

This is continuous (in fact any function from discrete top. is continuous since all subsets are open) and bijective, but not a homeomorphism ((\( \{0\} \subset X \) is not open, since any open ball around \( 0 \) contains \( \frac{1}{n} \) for large \( n \)).

\( f \) is a homeomorphism since every subset of \( N \) is open whereas not true for \( X \).

**A metric space is bounded** if \( \sup \{d(x, y) \mid x, y \in X\} < \infty \).

This is not a topological property! For example:

\[ f: \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}, \quad f(x) = \tan x \]

This is a continuous bijection, and \( f^{-1} = \text{arctan} \) is continuous as well, so \( \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \) is homeomorphic to \( \mathbb{R} \). (or in fact, any open interval).

**Def.** \( f: Y \to X \) continuous injective map, then \( f \) is an **embedding** if the map \( Y \to f(Y) \) is a homeomorphism (with subspace topology on \( f(Y) \subset X \)).

**Ex.** \( f: N \to \mathbb{R} \) \( f(0) = 0 \), \( f(n) = \frac{1}{n} \) for \( n \geq 1 \) is not an embedding.

\( Y \subset X \) with subspace topology, the inclusion \( i: Y \to X \) is an embedding.
Beware: in differential topology the notion of embedding is different, because one considers spaces not just topologically but with a smooth structure.

\[ \mathbb{R} \rightarrow \mathbb{R}^2 \\
\alpha \mapsto (x^2, x^3) \rightarrow \square \hspace{1cm} \text{is a topological embedding but not a smooth embedding} \]

---

Closed sets & limit points (Murphree §17)

Recall: a subset \( A \) of a topological space \( X \) is closed if \( X \setminus A \) is open.

(sets can be both closed & open, eg. \( \emptyset \) and \( X \), or neither.)

Def: \( A \subseteq X \) any subset

1) the closure of \( A \), \( \overline{A} = \) smallest closed set containing \( A \)
\[ = \bigcap_{F \supseteq A, F \text{ closed}} F \text{ (closed since it's an intersection of closed sets)} \]

2) the interior of \( A \), \( \text{int}(A) = \) largest open set contained in \( A \)
\[ = \bigcup_{\text{open} \subseteq A} U \text{ (open)} \]

3) the boundary of \( A \) is \( \partial A = \overline{A} \setminus \text{int}(A) \) (or \( \text{bd}(A) \))

Ex: \( A = [0,1) \subseteq \mathbb{R} \Rightarrow \overline{A} = [0,1] \), \( \text{int}(A) = (0,1) \), \( \partial A = \{0,1\} \)

Rmk: \( A \) is closed iff \( \overline{A} = A \), open iff \( \text{int}(A) = A \).

Def: say \( A \) is dense if \( \overline{A} = X \).

Ex: \( \mathbb{Q} \subseteq \mathbb{R} \) is dense in \( \mathbb{R} \).

Indeed, assume not, then \( \exists x \in X \setminus \overline{Q} \) which is open
\[ \Rightarrow \exists \epsilon > 0 \text{ s.t. } x \in (x-\epsilon, x+\epsilon) \subseteq X \setminus \overline{Q} \subseteq \mathbb{R} \setminus \overline{Q} \] but \( \exists \text{rationals in } (x, x+\epsilon) \).

Def: \( U \subseteq X \) is a neighborhood of \( x \in X \) if \( x \in U \) and \( U \) is open.

Def: \( x \in X \) is a limit point of \( A \subseteq X \) if, for every neighborhood \( U \) of \( x \),
\[ U \cap (A \setminus \{x\}) \neq \emptyset. \]

Ex: \( 1 \) is a limit point of \( (0,1) \) of \( [0,1] \).

\( 0 \) is not a limit point of \( \{\frac{1}{n}, n \geq 1\} \cup \{0\} \), but \( 0 \) is.
Then, \( \overline{A} = A \cup \{ \text{limit points of } A \} \).

**Proof:**
- Suppose \( x \notin A \) and \( x \) is not a limit point. Then \( \exists U \) neighborhood of \( x \) st. \( U \cap A = \emptyset \).
  
  Hence \( A \subseteq X - U \) which is closed \( \Rightarrow \overline{A} = \cap (\text{closed sets containing } A) \subseteq X - U \).
  
  So \( U \cap \overline{A} = \emptyset \), hence \( x \notin \overline{A} \).

- Conversely, suppose \( x \notin \overline{A} \). Then \( U = X - \overline{A} \) is an open neighborhood of \( x \) disjoint from \( A \), so \( x \) is not a limit point of \( A \) (nor in \( A \)). \( \square \).

**Corollary:** \( x \in \overline{A} \) iff \( \forall U \text{ neighborhood of } x, \ U \cap A \neq \emptyset \). (Proof: consider separately cases \( x \in A \), \( x \notin A \)).
Recall:\n\textbf{Def.} \quad x \in X \text{ is a limit point of } A \subset X \text{ if, for every neighborhood } U \text{ of } x, \quad U \cap (A - \{x\}) \neq \emptyset.

Then:\n\overline{A} = \bigcap_{F \supseteq A, \text{F closed}} F = A \cup \{\text{limit points of } A\}.

\textbf{Corollary.} \quad x \in \overline{A} \iff \forall U \text{ neighborhood of } x, \quad U \cap A \neq \emptyset.

---

\textbf{Limits of Sequences}:

\[ X \text{ top space: Then we say a sequence } x_1, x_2, \ldots \text{ converges to } x \text{ if, for every neighborhood } U \text{ of } x, \quad \exists N \text{ s.t. } n \geq N \implies x_n \in U. \]

Rmk: enough to check this for a basis of neighborhoods of \( x \), i.e. a family of neighborhoods s.t.

\( \forall \) neighborhood of \( x \) contains one of them.

e.g. in a metric space, balls \( B_r(x), r > 0 \), or even the balls \( B_{1/n}(x) \).

\( \rightarrow \) taking base, recover usual notion:

\[ x_n \to x \iff \forall r > 0 \quad \exists N \text{ s.t. } n \geq N \implies x_n \in B_r(x). \]

\textbf{Fact.} \quad \text{If } \exists \text{ sequence } x_n \text{ in } A \text{ with } x_n \to x \text{ in } A, \text{ and } x_n \to \text{ then } x \text{ is a limit point of } A.

\( \text{(indeed: } U \text{ neighborhood of } x, \quad U \cap (A - \{x\}) \supseteq x_n \text{ for all sufficiently large } n! \)\)

Conversely, in a metric space, if \( x \) is a limit point of \( A \subset X \),

\[ \forall r > 0 \quad \exists x_n \in \overline{B_r(x)} \cap A \text{ with } x_n \not\in x \]

\& hence \( x \) is the limit of a sequence \( \{x_n\} \) in \( A \) with \( x_n \not\in U \).

This holds more generally in spaces whose points have countable bases of neighborhoods

\[ U_1 \supset U_2 \supset \ldots \text{ (i.e. } U_1 \text{ includes } U_2, U_3, \ldots \text{ s.t. } \forall U \exists x, \quad \exists n \text{ s.t. } x \in U, x \in U_n \text{, but not in arbitrary topological space!} \]

\textbf{Ex.} \quad \text{Let } X = \mathbb{R} \text{ with topology } \mathcal{Y} = \{U \subset \mathbb{R} / U = \emptyset \text{ or } \mathbb{R} - U \text{ is countable}\}.

\( \text{(check this satisfies the axioms). Let } \ A = (0,1). \ \text{Then } \overline{A} = \mathbb{R} \)

\( \text{(indeed: closed } \Rightarrow \text{ countable or all } \mathbb{R}, \text{ so smallest closed set } \\
\text{containing } (0,1) \text{ is } \mathbb{R}. \)

\( \text{hence } 2 \text{ is a limit point of } A! \)

\text{But there is no sequence } a_n \in A \text{ s.t. } a_n \to 2, \text{ since the complement of any}

\text{sequence in } A \text{ is open, hence a neighborhood of } 2 \text{ containing no } a_n\'s.

\( \text{(in fact for a seq. to converge in this topology it must be constant after finitely many terms).} \)}
Hausdorff spaces:

Recall: in a metric space, a sequence converges to at most one limit. This is not true in an arbitrary topological space!

**Ex:** \( X = \mathbb{R} \) with finite complement topology (\( \text{open} = \emptyset \) and \( \mathbb{R} - \{ \text{finite sets} \} \)).

Let \( a_1, a_2, \ldots \) be a sequence in \( X \) with all \( a_i \) distinct.

Then \( \forall x \in X \), every neighborhood \( U \ni x \) has finite complement, hence contains all but finitely many of the \( a_i \), hence \( \exists \text{N s.t. } a_n \notin U \forall n \geq N \).

Thus, the sequence converges to every point of \( X \)!

To avoid such pathological behavior:

**Def:** A top-space is **Hausdorff** if \( \forall x_1 \neq x_2 \in X \), \( \exists \) neighborhoods \( U_1 \ni x_1 \), \( U_2 \ni x_2 \) s.t. \( U_1 \cap U_2 = \emptyset \).

**Ex:**
1) any metric space is Hausdorff:
   - Given \( x_1 \neq x_2 \), choose \( 0 < \varepsilon < \frac{1}{2} \text{d}(x_1, x_2) \)
   - Then \( U_i = B_\varepsilon(x_i) \) disjoint neighborhoods of \( x_i \).

2) the finite complement topology on \( \mathbb{R} \) is not Hausdorff, since any two non-empty open sets intersect (in infinitely many points).

3) the discrete topology is always Hausdorff \( (U_i = \{ x_i \} \) disjoint neighborhoods of \( x_i \))

**Thm:** If \( X \) is Hausdorff then every sequence in \( X \) converges to at most one limit.

**Proof:** Assume \( x_1, x_2, \ldots \) converges to \( x \in X \), and let \( y \neq x \).

Choose \( U_x \ni x \), \( U_y \ni y \) disjoint neighborhoods.

Since \( x_n \to x \), \( \exists \text{N s.t. } \forall n \geq N \) \( x_n \notin U_y \). Hence \( x_n \notin U_y \) for \( n \geq N \), so the sequence doesn't converge to \( y \).

\( \Box \)

Remark: there are several flavors of **separation axioms**, besides the notion of Hausdorff-ness:

From weakest to strongest:
- \( X \) is **T₀** if \( \forall x \neq y \), there exists an open set containing one but not the other (but not necessarily vice-versa).
- \( X \) is **T₁** if \( \forall x \neq y \), \( \exists \) neighborhood \( U \ni y \) which doesn't contain \( x \), (\( \iff \forall x, \{ x \} \) is closed) (indeed; consider \( \mathbb{R} - \{ x \} \) vs. \( U_y \ni y \)).
\( T_1 \) Hausdorff: \( \forall x \neq y, \exists \text{ neighborhoods } U \ni x, V \ni y \text{ st. } U \cap V = \emptyset \)

\( T_2 = \text{regular}: \quad T_1 + \forall x \in X, \forall A \text{ closed st. } x \notin A, \exists \text{ open } U \ni x, V \ni A, U \cap V = \emptyset \)

\( T_3 = \text{normal}: \quad T_2 + \forall A, B \subseteq X \text{ closed & disjoint, } \exists \text{ open } U \supseteq A, V \supseteq B, U \cap V = \emptyset \)

**Ex:** \( R \) with the first coordinate topology is \( T_2 \) (\( \mathbb{R} \setminus \{0\} \) open in \( X \)) but not Hausdorff (as seen above).

**Ex:** \( \mathbb{R} \) is normal; \( \mathbb{R}^e \subset \mathbb{R} \) is regular but not normal. (Munkres end of §31).

The motivation for studying normal & regular spaces comes from the question of metrizability, i.e. which topologies are actually metric space topologies.

**Theorem:** Every metric space is normal

**Proof:** Let \( A, B \subseteq X \text{ closed & disjoint.} \)
\[
\forall a \in A \quad \exists \epsilon_a > 0 \text{ st. } B(a, \epsilon_a) \subseteq X - B.
\]
\[
\forall b \in B \quad \exists \epsilon_b > 0 \text{ st. } B(b, \epsilon_b) \subseteq X - A.
\]

Observe: \( d(a, b) = \max(\epsilon_a, \epsilon_b) \geq \frac{\epsilon_a + \epsilon_b}{2} \) \( \forall a \in A \quad \forall b \in B, \text{ hence } B(a, \frac{\epsilon_a}{2}) \cap B(b, \frac{\epsilon_b}{2}) = \emptyset. \)

This implies: \( U = \bigcup_{a \in A} B(a, \frac{\epsilon_a}{2}) \) and \( V = \bigcup_{b \in B} B(b, \frac{\epsilon_b}{2}) \) are disjoint (and clearly open, contain \( A \cap B \)).

**Urysohn metrization theorem:** Every regular space with a countable basis is metrizable. (i.e. \( \exists \) metric inducing the topology).

---

**Topology on products (§13)**

Given an index set \( I \), and topological spaces \( X_i; i \in I \), consider the product set \( X = \prod_{i \in I} X_i = \{ (a_i)_{i \in I} \mid a_i \in X_i; \forall i \in I \} \)

Natural topology on \( X \)?

First idea: the box topology

**Def:** the box topology on \( \prod_{i \in I} X_i \) has basis \( \{ \prod_{i \in I} U_i \mid U_i \subseteq X_i \text{ open } \forall i \} \)

(this is a basis: box \( \cap \) box = box, since \((\prod_{i \in I} U_i) \cap (\prod_{i \in I} V_i) = \prod_{i \in I} (U_i \cap V_i))\)
This is a natural definition, but has unexpected properties.

**Example:** Consider the diagonal map \( \Delta: \mathbb{R} \to \mathbb{R}^\omega = \mathbb{R}^\mathbb{N} (= \mathbb{R}_0 \times \mathbb{R}_1 \times \mathbb{R}_2 \times \ldots) \)

\[ \Delta(x) = (x, x, x, \ldots) \]

For finite products, with product topology, \( \Delta: \mathbb{R} \to \mathbb{R}^n \) is continuous (in fact, an embedding).

But, giving \( \mathbb{R}^\mathbb{N} \) the box topology, \( \Delta \) is not continuous!!

Indeed, let \( U = (-1, 1) \times (-1/2, 1/2) \times (-1/3, 1/3) \times \ldots \) open in box topology.

\[ \Delta^{-1}(U) = \bigcap_{n>1} \left( \left(-\frac{1}{n}, \frac{1}{n}\right) \right) = \emptyset \] not open in \( \mathbb{R}^\mathbb{N} \).

**Better:** The product topology

Define: the product topology on \( X = \prod X_i \) has basis

\[ \left\{ \prod U_i \mid U_i \subset X_i \text{ open, and } U_i = X_i \text{ for all but finitely many } i \right\} \]

(check this is a basis!)

(This is the same as the box topology if \( I \) is finite; for infinite \( I \) this is coarser than the box topology)

Unless otherwise specified, the product topology is the one we'll use on \( \prod X_i \).

**Theorem:** \( f: \mathbb{Z} \to \prod X_i \) is continuous \( \iff \) each component \( f_i: \mathbb{Z} \to X_i \) is continuous.

\( \mathbb{Z} \mapsto \left( f_i(z) \right)_{i \in I} \) product top

**Ex:** This now implies the diagonal map \( \Delta: \mathbb{R} \to \mathbb{R}^\mathbb{N} \) is continuous, since each \( \Delta_i = \text{identity} \).

**Proof:** Assume \( f \) is continuous. The component maps are \( f_i = p_i \circ f \) where

\( p_i: X \to X_i \) is the projection to the \( i \)-th factor.

\( p_i \) is continuous since \( U \subset X \) open, \( p_i^{-1}(U) = \text{product of } \{ X_j \text{ for } j \neq i \text{ open} \} \)

Hence \( f_i = p_i \circ f \) is continuous (composition of 2 continuous functions).

Conversely, assume all \( f_i \) are continuous, and consider basis element

\( \prod U_i \subset X \) where \( U_i = X_i \) for all but finitely many \( i \).

Then \( f^{-1}(\prod U_i) = \left\{ z \in \mathbb{Z} \mid \left( f_i(z) \right)_{i \in I} \in \prod U_i \right\} = \bigcap_{i \in I} f_i^{-1}(U_i) \)

Each \( f_i^{-1}(U_i) \subset \mathbb{Z} \) is open, and all but finitely many are \( f_i^{-1}(X_i) = \mathbb{Z} \), so can be omitted from the intersection.

So \( f^{-1}(\prod U_i) \) is the intersection of finitely many open sets in \( \mathbb{Z} \), hence open. \( \square \)
Recall: on the product \( X = \prod_{i \in I} X_i = \{ (x_i)_{i \in I} / x_i \in X; \forall i \in I \} \) of top. spaces \( X_i, i \in I \), the most obvious topology is the box topology, with basis \( \{ \prod_{i \in I} U_i / U_i \subset X_i \text{ open}, \forall i \} \), which is not as well-behaved as the product topology, which has basis \( \{ \prod_{i \in I} U_i / U_i \subset X_i \text{ open, and } U_i = X_i \text{ for all but finitely many } i \} \).

**Theorem:** \( f : \mathbb{Z} \to X = \prod X_i \text{ is continuous } \iff \text{ each component } f_i : \mathbb{Z} \to X_i \text{ is continuous}. \)

**Proof:** Assume \( f \) is continuous. The component maps are \( f_i = p_i \circ f \) where \( p_i : X \to X_i \) is the projection to the \( i \)th factor. \( p_i \) is continuous since \( U U \subset X_i \text{ open, } p_i^{-1}(U) = \text{ product of } \{ X_j \text{ for } j \neq i \text{ open}. \) Hence \( f_i = p_i \circ f \) is continuous (composition of 2 continuous functions).

Conversely, assume all \( f_i \) are continuous, and consider basis element \( \prod U_i \subset X \) where \( U_i = X_i \) for all but finitely many \( i \).

Then \( f^{-1}(\prod U_i) = \{ z \in \mathbb{Z} / (f_i(z))_{i \in I} \subset \prod U_i \} = \bigcap_{i \in I} f_i^{-1}(U_i) \)

Each \( f_i^{-1}(U_i) \subset \mathbb{Z} \) is open, and all but finitely many are \( f_i^{-1}(X_i) = \mathbb{Z} \), so can be omitted from the intersection.

So \( f^{-1}(\prod U_i) \) is the intersection of finitely many open sets in \( \mathbb{Z} \), hence open. \( \square \)

On products of metric spaces, there is another natural topology, finer than the product but coarser than the box topology — the uniform topology.

This works similarly to the construction of \( d_{\infty}(x,y) = \sup (|y_i - x_i|) \text{ on } \mathbb{R}^n \), but for an infinite product the sup might be infinite. So:

1. **First step:** can replace the metric on \( (X,d) \) by \( \overline{d}(x,y) = \min(d(x,y),1) \), this is still a metric (check!) and induces the same topology as \( d \).

Indeed: basis for \( \overline{d} = \text{balls of radius } r \leq 1 \) (for \( d \) on \( \mathbb{R}^n \) no difference here) + all \( X \). This is also a basis for \( d \) (balls of radius \( > 1 \) not needed!)

\( U \subset (X,d) \) is open iff \( \forall x \in U, \exists r \in (0,1) \text{ s.t. } B_r(x) \subset U \).
Now, given a collection of metric spaces \((X_i, d_i)_{i \in I}\) replace each \(d_i\) by bounded metric \(\overline{d}_i\), and define a metric \(\overline{d}_{\infty}\) on \(\prod X_i\) by

\[
\overline{d}_{\infty}(x, y) = \sup \{ \overline{d}_i(x_i, y_i) \mid i \in I \} = \begin{cases} 
\sup \{ d_i(x_i, y_i) \} & \text{if } i \leq 1
\end{cases}
\]

This is called the uniform metric and induces the uniform topology.

**Ex: on \(\mathbb{R}^\mathbb{N} = \{\text{functions } I \to \mathbb{R}\}\), (with usual distance on \(\mathbb{R}\)), this is

\[
\overline{d}_{\infty}(f, g) = \sup \{ |f(i) - g(i)| \mid i \in I \}
\]

so \(f_n \to f \iff \overline{d}_{\infty}(f_n, f) \to 0 \iff \sup \{|f_n - f|\} \to 0\)

ie. uniform convergence!

Here the name.

Q:\ How does this compare with box & product topologies? First need to understand balls!

* If \(I\) is finite, then open balls of radius \(r \leq 1\) for \(d_{\infty}\) are products of open balls

\[
B_{\overline{d}_{\infty}}^r(x) = \prod \overline{B}_{d_i}^r(x_i)
\]

e.g. in \(\mathbb{R}^n\), \((x_1 - r, x_1 + r) \times \ldots \times (x_n - r, x_n + r)\).

* What if \(I\) is infinite?

Observe: \(U = (-1, 1) \times (-1, 1) \times \ldots = (-1, 1)^\mathbb{N} \subset \mathbb{R}^\mathbb{N}\)

is actually not the ball of radius 1 around \((0, 0, 0, \ldots)\) for \(d_{\infty}\)!

Indeed, \(x = (0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots)\) \((n^{th}\text{ entry is } \frac{n}{n+1})\) is in \(U\), but

\[
\overline{d}_{\infty}(0, x) = \sup \{ \frac{n}{n+1} \mid n \in \mathbb{N}^+ \} = 1 \not\leq 1,
\]

so \(x \notin B_1(0)\).

Moreover, \(U\) is not open! since \(x \in U\), but for any \(\varepsilon > 0\), \(B_{\varepsilon}(x) \not\subset U\)

since some coordinates of \(x\) are within \(\varepsilon\) of \(1\).

* What are the balls, then? (of radius \(r \leq 1\) - if \(r > 1\) get all of \(\prod X_i\))

  - If \(r' \leq r\) then \(U_{r', r}(x) = \prod \overline{B}_{d_i}^{r'}(x_i)\) (product of balls of radius \(r'\) in \(X_i\))

    is contained in the ball of radius \(r\).

    (Indeed, \(d_i(x_i, y_i) < r' \iff i \in I\Rightarrow \overline{d}_{\infty}(x, y) = \sup d_i(x_i, y_i) \leq r' < r\).

  - Conversely, if \(\overline{d}_{\infty}(x, y) < r\) then, choosing \(r' > r\), \(d_i(x_i, y_i) < r'\) \forall i \in I\) so \(y \in U_{r', r}(x)\).

Hence: \(B_r(x) = U_{r', r}(x)\), \(0 < r' < r\) (\(\neq\) if \(I\) infinite).
Theorem: The uniform topology on $\Pi (X_i, d_i)$ is finer than the product topology.

Proof: Let $x = (x_i)_{i\in I} \in \Pi X_i$, and $\Pi U_i \ni x$ a basis element in the product topology.

1) For each $i$, $U_i \ni x_i$ open so $\exists \varepsilon_i > 0$ s.t. $B_{\varepsilon_i}(x_i) \subset U_i$.
   
   Thus we can assume $\varepsilon = \inf \{\varepsilon_i\}$ satisfies $0 < \varepsilon \leq 1$.

   Now $B_{\varepsilon}(x) \subset U_{x}(x) = \prod_{i} B_{\varepsilon_i}(x_i) \subset \prod_{i} U_{i}$.

   This proves $\Pi U_i$ is open in the uniform topology; hence $\Pi_{\text{uniform}} \subset \Pi_{\text{box}}$.

2) Let $B = B_{\varepsilon}(x) = \bigcup_{0<\varepsilon<\varepsilon} U_{x}(x)$ ball for uniform metric.

   Then $\forall y \in B$, $\exists \varepsilon > 0$ s.t. $y \in U_{x}(x) = \prod_{i} B_{\varepsilon}(x_i) \subset B$.

   Hence $B$ is open in box topology, and $\Pi_{\text{uniform}} \subset \Pi_{\text{box}}$.

Note: for finite products, box and product topologies coincide, so the theorem then implies that the uniform topology also coincide with those. However if $I$ is infinite then the three topologies are all different.

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**Connected space** (Munkres 23-24)

A topological space $X$ is *connected* if it cannot be written as $X = U \cup V$ where $U, V$ are disjoint nonempty open sets.

**Ex:** $[0, 1]$ (with standard topology as subset of $\mathbb{R}$) is connected.

Indeed, assume $[0, 1] = U \cup V$ separation. Without loss of generality, $0 \in U$.

Let $a = \sup \{x \in [0, 1] \mid [0, x) \subset U\}.$

- $0 \in U$, $U_{\text{open}} \Rightarrow [0, \varepsilon) \subset U$ for some $\varepsilon > 0 \Rightarrow a > 0.$
- Can't have $a \in V$; indeed $V$ is open so this would imply $(a-\varepsilon, a) \subset V$ for some $\varepsilon > 0$, and hence $[0, x) \notin U$ for $x > a-\varepsilon$, hence $\sup \{x \mid x \in (0, x) \} \leq a-\varepsilon$, contradiction. So $a \in U$.

- But if $a < 1$, $U_{\text{open}} \Rightarrow \exists \varepsilon > 0$ s.t. $(a-\varepsilon, a+\varepsilon) \subset U$, and by def. of $a$, $\exists x > a-\varepsilon$ s.t. $(0, x) \subset U$. Hence $[0, a+\varepsilon) \subset U$, contradicting def. of $a$. 

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Hence $a = 1$, and since $U$ is open, $\exists \varepsilon > 0 \text{ s.t. } (1-\varepsilon,1) \subseteq U$, & by def. of $a$, $\exists x > 1-\varepsilon \text{ s.t. } [0,x) \subseteq U$, hence $U = [0,1]$, and $V = \emptyset$. Contradiction. \(\square\)

**Ex.** $[0,1) \cup (1,2]$ is not connected, since $[0,1)$ and $(1,2]$ are open in subspace topology & provide a separation.

More generally, $x, y \in \mathbb{R}, x, z \in A, y \notin A \Rightarrow A$ disconnected. $(\cap \mathbb{A} (-\infty, y)) \cup (\cap \mathbb{A} (y, \infty))$. 
Recall: A topological space $X$ is \textit{connected} if it cannot be written as $X = U \cup V$ where $U, V$ are disjoint nonempty open sets. (Such a decomposition is called a \textit{separation} of $X$.)

\textbf{Ex.} $[0,1]$ (with standard topology as subset of $\mathbb{R}$) is connected.

\textbf{Ex.} $[0,1) \cup (1,2]$ is not connected, since $[0,1)$ and $(1,2]$ are open in subspace topology and provide a separation. More generally, $x \leq y < z \in \mathbb{R}$, $x, z \notin A, y \in A \Rightarrow A$ disconnected.

\textbf{Ex.} $\mathbb{R}^e = (-\infty, 0) \cup [0, \infty)$ both open, so $\mathbb{R}^e$ is not connected.

In fact, any subset of $\mathbb{R}^e$ containing more than one point is disconnected. Say $\mathbb{R}^e$ is \textit{totally disconnected}. (Prove this!) \hfill (Ps. Exercise!)

\textbf{Ex.} $(\mathbb{R}, \text{finite compl. top.})$ is connected — since any two nonempty open subsets intersect, no separation.

\textbf{Fact.} $A, B \subseteq X$ connected (for subspace top.) $\Rightarrow$ $A \cap B$ connected. \hfill \begin{tikzpicture}
    \draw[fill=blue!20] (0,0) circle (1);
    \draw[fill=red!20] (1.5,0) circle (1);
    \node at (2,0) {\begin{tikzpicture}
        \draw[thick, fill=blue!20] (0,0) circle (0.5);
        \draw[thick, fill=red!20] (1.5,0) circle (0.5);
    \end{tikzpicture}};
    \node at (0.75,0.75) {A};
    \node at (1.75,0.75) {B};
    \node at (1,1) {$\subseteq \mathbb{R}^2$};
\end{tikzpicture}

But things are better for unions of connected sets as long as they overlap.

\textbf{Thm.} \begin{itemize}
    \item $A_i \subseteq X$ connected subspaces, all containing some point $p \in X$ (ie. $\forall A_i \neq \emptyset$).
    \item Then $Y = \bigcup A_i$ is connected.
\end{itemize}

\textbf{Pf.} Assume $Y = U \cup V$ disjoint, open in $Y$. Without loss of generality, $p \in U$.

Then $U \cap A_i$ and $V \cap A_i$ are disjoint, open in $A_i$. Since $A_i$ is connected and $p \in U \cap A_i$, must have $A_i \subseteq U \forall i$. Hence $Y = \bigcup A_i \subseteq U$ (and $V = \emptyset$).

So $Y$ is connected. \hfill $\square$

\textbf{Corollary.} $\mathbb{R}$ is connected; so are open, half-open, and closed intervals in $\mathbb{R}$.

\textbf{Pf.} We've seen $[0,1]$ is connected; so is $[a,b]$ (homeomorphic to $[0,1]$).

Hence $U \cup \bigcup_{n=1}^{\infty} [-n, n] = \mathbb{R}$ is connected (using the theorem: all $[a,n] \cap \mathbb{R}$ contain 0) & similarly for open and half-open intervals. \hfill $\square$

\textbf{Thm.} \begin{itemize}
    \item $f : X \to Y$ continuous, $X$ connected $\Rightarrow f(X) \subseteq Y$ is connected.
\end{itemize}

\textbf{Pf.} Since the map $X \to f(X)$ is continuous (for subspace top.), can assume $Y = f(X)$.

Now if $Y = U \cup V$ is a separation of $Y$, then $X = f^{-1}(U) \cup f^{-1}(V)$.

$f^{-1}(U)$ & $f^{-1}(V)$ open (f continuous), nonempty ($U, V$ nonempty & f auto), disjoint, contradiction. $\square$
Intermediate Value Theorem

**Theorem:** If $X$ is a connected topological space, and $f: X \rightarrow \mathbb{R}$ is continuous, and $a, b \in X$ with $f(a) < r < f(b)$, then there exists $c \in X$ such that $f(c) = r$.

**Proof:** Since $X$ is connected, so is $f(X)$. If $r \notin f(X)$, then $U = (-\infty, r) \cap f(X)$ and $V = (r, \infty) \cap f(X)$ give a separation of $f(X)$ (one contains $f(a)$ and the other contains $f(b)$) — contradiction. So $r \in f(X)$. \( \square \)

**Corollary:** Products of connected spaces

If $X, Y$ are connected, then $X \times Y$ is connected.

**Proof:** Suppose $X \times Y = U \cup V$, where $U$ and $V$ are non-empty open sets. We will show $U = X \times Y$.

Since $X \times \{b\}$ is connected, $X \times \{b\} = (U \cap (X \times \{b\})) \cup (V \cap (X \times \{b\}))$ is a separation of $X \times \{b\}$, so $X \times \{b\} \subseteq U$ (and $V \cap (X \times \{b\}) = \emptyset$).

Next, let $x \in X$. If $x \times Y$ is connected, then $x \times Y = (U \cap (x \times Y)) \cup (V \cap (x \times Y))$ is a separation of $x \times Y$, so $x \times Y \subseteq U$.

So $U = \bigcup_{x \in X} \{x\} \times Y = X \times Y$ and $V = \emptyset$. Hence $X \times Y$ is connected. \( \square \)

**Corollary:** The product of connected spaces is connected.

This remains true for infinite products.

**Theorem:** If $(X_i)_{i \in I}$ is a collection of connected spaces, then $\prod_{i \in I} X_i$ with the product topology is connected.

**Exercise:** (Key point: $(a_i)_{i \in I}$ is open $\Rightarrow \{a_{i \in I} \times \prod_{i \in I \setminus \{i \in I\}} X_i \subseteq U$ so only finitely many steps remaining in above argument).

This is not true in the box topology, or in the uniform topology.

**Example:** $X = \mathbb{R}^n$ with box topology or uniform topology.

Let $U$ be set of bounded sequences, i.e. $(a_1, a_2, \ldots) \in U$ if $M > 0$ such that $|a_i| \leq M$.

$V$ is set of unbounded sequences. Clearly $U \cup V = X$.

Given $a = (a_1, a_2, \ldots) \in X$, $B_a = \prod_{i \in \mathbb{N}} (a_{i-1}, a_{i+1})$ is a neighborhood of $a$ in box topology (contains ball in uniform topology); and every elt of $B_a$ is bounded if $a$ is bounded unbounded.

So $U$ and $V$ are open (in both topologies).
Path-connected spaces:

**Def:** If $X$ is a top space, $x,y \in X$, a **path** from $x$ to $y$ is a continuous map $f : [a,b] \to X$ s.t. $f(a) = x$ and $f(b) = y$.


**Def:** $X$ is **path-connected** if every pair of points in $X$ can be joined by a path.

**Note:** The relation $x \sim y$ is an equivalence relation, i.e.

1. $x \sim x$ (constant path $f(t) = x$)
2. $x \sim y \implies y \sim x$ (backward path $f(-t)$)
3. $x \sim y$ and $y \sim z \implies x \sim z$ (concatenate paths: $f = [f_1(t) : t \in [a,c]] \cup [f_2(t) : t \in [c,b]]$)

The equivalence classes are called the **path components** of $X$.

(Theorem: if $X$ is path connected then $X$ is connected.

**Proof:** Assume $X = U \cup V$ disjoint open, $x \in U$.

Then $f([a,b])$ is connected, so $f([a,b]) \subseteq U$.

Hence $f(b) = y \in U$.

Thus $y \in U$ for any other point $y \in X$, $x \in X$.

The converse isn't true!

**Ex:** the "topologist's sine curve"; let $S = \{(x,y) \mid y = \sin \left(\frac{1}{x}\right), x > 0\} \cup \{(0,0)\} \subseteq \mathbb{R}^2$.

The "main" part $S_0$ is connected, since it's the image of $\mathbb{R}_+$(connected) under the continuous map $x \mapsto (x, \sin \frac{1}{x})$.

$(0,0)$ is a limit point of $S_0$ (limit of sequence $(\frac{1}{n\pi}, 0)$).

Hence $S$ is connected: indeed, if $S = U \cup V$, $(0,0) \in U \implies U$ contains a neighborhood of $(0,0)$ hence $S_0 \cap U \neq \emptyset$, but $S_0 = (S_0 \cap U) \cup (S_0 \cap V)$ and $S_0$ is connected $\implies S_0 \subseteq U$ and so $U = S$, $V = \emptyset$. 
However, $S$ is not path connected because there is no path connecting $(\frac{1}{\pi}, 0)$ to $(0,0)$. Indeed, if we use such a path $f:[a,b]\times S$, then by intermediate value theorem, $x$ coordinate would need to take all values between $\frac{1}{\pi}$ and 0, hence

\[ \exists t_1, t_2, \ldots \in [a,b] \text{ s.t. } f(t_n) = \left(\frac{1}{2\pi n + \frac{\pi}{2}}, 1\right) \to (0,1) \]

Can find a convergent subsequence of $\{t_n\}$ which goes to some limit too (e.g. $\lim \inf t_n$). (see later, sequential compactness) and then by continuity $f(\lim t_n) = \lim f(t_n) = (0,1) \notin S$, contradiction.

However, for "well-behaved" spaces, connectedness is the same as path connectedness.

Then: $A \subseteq \mathbb{R}^n$ open $\Rightarrow$ $A$ is connected iff $A$ is path connected.

**Pf:** already seen: path connected $\Rightarrow$ connected.

Conversely, assume $A$ open in $\mathbb{R}^n$: then the path components of $A$ are open.

Indeed, if $x \in A$ then $\exists r > 0$ s.t. $B_r(x) \subseteq A$, and any two points of $B_r(x)$ can be connected inside $A$ by a straight line segment.

So all of $B_r(x)$ is in the same path component as $x$.

(If $A$ path connected then get to any other point by adding a line segment to the path — and conversely).

Next, let $U \subseteq A$ be the path component containing some $a \in A$,

$V =$ union of all other path components.

Then $A = U \cup V$, $U, V$ open $\Rightarrow$ if $A$ is connected then $U = A$, $V = \emptyset$, $U \ni a$ so $A$ is path-connected.
1 Compactness

Today we’re going to start talking about the least intuitive of the topics so far - it’s hard to visualize what it means for a space to be compact, but we’ll try to give you useful heuristics and help you to understand what it means for a space to be compact.

In a course on Analysis, we learn that a closed interval \([a,b] \subseteq \mathbb{R}\) is compact, and that more generally any set in \(\mathbb{R}^n\) that’s closed and bounded is compact. Unfortunately, there’s not a natural definition of "bounded" for a general topological space, so we’ll need something else. But why do we care about compactness? Compact spaces enjoy some nice properties:

1. generalize metric space notion of "boundedness"

2. Any continuous map \(f : K \to \mathbb{R}\) achieves a maximum and a minimum.\(^1\)

**Definition 1.** Let \(X\) a topological space. A collection of open sets \(\{U_i\}_{i \in I}\) is an open cover if \(\bigcup U_i = X\).

**Definition 2.** \(X\) is compact if every open cover contains a finite subcollection that also covers \(X\). This finite subcollection is called a finite subcover.

This is a tricky property to get a handle on - in order to show that a space is compact, you have to check every open cover has a finite subcover. On the other hand, we need only to exhibit one open cover without a finite subcover to show that a space is not compact.

**Example 3.** \(\mathbb{R}\) is not compact. Take the cover \(\mathbb{R} = \bigcup(n, n + 2)\). Removing any of these sets excludes the integer \(n + 1\). This is good news, because it gives us hope that our general definition of compactness agrees with the analysis definition. \(\mathbb{R}\) is not closed and bounded, so it should not be compact.

**Example 4.** \((0, 1]\) is not compact. The cover \(\bigcup(\frac{1}{n}, 1]\) has no finite subcover.

**Example 5.** Let \(X = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{Z}^+\}\). \(X\) is compact. This is because there must be some element in the open cover which contains \(\{0\}\). The sequence \(\frac{1}{n}\) converges to 0, so any open set containing \(\{0\}\) must contain all but finitely many of the points \(\frac{1}{n}\). For each of the remaining points which are excluded, we can take one open set around each of them. This finite collection covers \(X\), so \(X\) is compact.

1.1 Basic results about compact spaces

**Theorem 6.** If \(A\) is compact and \(f : A \to X\) continuous, then \(f(A)\) is compact.\(^2\)

**Proof.** Let \(\bigcup U_i\) be an open cover of \(f(A)\). The preimages \(f^{-1}(U_i)\) is an open cover of \(A\). \(A\) is compact, so this has a finite subcover \(\bigcup_{j \in J} f^{-1}(U_j)\). \(U_j = f(f^{-1}(U_j))\), so these sets must cover \(f(A)\), since their preimages cover \(X\). \(U_j\) is a finite subcover of the original cover, so \(f(A)\) is compact. \(\Box\)

**Theorem 7.** \([0, 1]\) is compact.

\(^1\)This happens a lot, in particular with distance functions. A distance function from a compact space to \(\mathbb{R}\) always achieves a max and min, and this fact is quite powerful.

\(^2\)I is arbitrary - it need not be finite or countable

\(^3\)Taking \(X = \mathbb{R}\) gives the statement about continuous functions on compact spaces attaining extreme
Proof. Let \( \{U_i\} \) be an open cover of \([0,1]\). Let \( A = \{ x \in [0,1] \mid \exists \text{ a finite subcover of } [0,x] \} \). This is nonempty, because we know it contains 0. We will show that \( A \) is both open and closed. \([0,1]\) is connected, so this will imply that \( A = [0,1] \).

If a finite subcover works for \([0,x]\), then \( x \in U_i \) for some \( i \). \( U_i \) is open, so there exists \( B_r(x) \subseteq U_i \) for some \( r > 0 \). Then \( x \in B_r(x) \subseteq A \), so \( A \) is open.

To show that \( A \) is closed, suppose \( x \) is a limit point of \( A \). Then \( x \in U_i \) for some \( i \), so \( x \in B_r(x) \subseteq U_i \) for some \( r \). The interval \([0, x - \frac{r}{2}]\) admits a finite cover. If we just include \( U_i \), then there is a finite subcover for \([0,x]\) too, and \( x \in A \). \( A \) contains all of its limit points, and so is closed.

In the \( \mathbb{R}^n \) case, from analysis, we know that compact spaces are the ones that are closed and bounded. There’s no general topological notion of boundedness, but we can ask how compact spaces are related to closed sets.

**Theorem 8.** If \( X \) is compact, then any closed subspace \( A \subseteq X \) is compact.

**Proof.** Let \( A \) be an open covering of \( A \) by sets open in \( X \). Then \( A \cup \{X \setminus A\} \) is an open cover of \( X \). Take a finite subcover of this cover, which exists because \( X \) is compact, to get a finite subcover of \( A \).

**Warning!** 1. This relied crucially on the fact that \( A \) was closed, so that \( X \setminus A \) is open. This is related to a common mistake in showing compactness - to show compactness we have to begin with an arbitrary open cover of our set, not one that has nice properties, such as also covering \( X \).

In a compact space, closed spaces are compact. Is the converse true? In the \( \mathbb{R}^n \) case, the answer is yes, but for more general topological spaces the answer is no.

**Example 9.** Let \( X \subseteq \mathbb{R} \) with the cofinite topology. \( X \) is always compact, because any open set contains all but finitely many points. But \( X \) might not be closed - any infinite set is not closed, though they are still compact.

However, the cofinite topology is in some sense "not nice", so there is hope that by imposing some "niceness" conditions, we can get this statement to be true, and more closely align with our intuition from the real case.

**Theorem 10.** Let \( X \) be a Hausdorff space. If \( K \subseteq X \) compact, then \( K \) is closed.

**Proof.** We will show that \( X \setminus K \) is open. Take \( x \in X \setminus K \) arbitrary\(^4\). For every \( y \in K \), there exist disjoint neighborhoods \( y \in U_y, x \in V_y \). Take one such neighborhood for each point \( y \in K \). \( K \) is compact, so only finitely many are required, say \( \{U_i\} \). Then take the intersection \( \bigcap_i V_i \). This is a finite intersection of open sets, so is open, and it doesn’t meet any of the points in \( K \), so \( X \setminus K \) is open.

**Theorem 11.** A continuous bijection \( f : X \rightarrow Y \) with \( X,Y \) compact Hausdorff is a homeomorphism.\(^6\)

**Proof.** We want to show that the image of closed sets are closed, as this will show that the inverse is continuous. Closed subsets of compact spaces are compact, and the image of a compact space is compact, and a compact subset of a Hausdorff space is closed, so \( f \) takes closed sets to closed sets, as desired. \( \square \)

\(^4\)there’s a picture here that makes this clear - you should try and draw it

\(^5\)If \( X \setminus K \) is empty, then \( K = X \) and is closed trivially

\(^6\)Nice properties like this are desirable, and in further applications of topology, such as in the theory of manifolds, we often restrict ourselves to spaces which are compact and Hausdorff.
Note that we can actually relax the assumptions on $X$ and $Y$ - we only require that $X$ is compact and that $Y$ is Hausdorff. This is a homeomorphism, so it will follow that $X$ and $Y$ are both compact and Hausdorff, but to use the theorem we only need to prove that $X$ is compact and that $Y$ is Hausdorff.

Note that this only works for compact Hausdorff spaces. Consider the bijection $f : [0, 1) \rightarrow S^1$ given by $x \mapsto e^{2\pi ix} = (\cos(2\pi x), \sin(2\pi x))$. This is a continuous bijection, but not a homeomorphism, since $S^1$ is compact, but $[0, 1)$ is not. Similarly, removing a point other than 0 from $[0, 1)$ leaves it disconnected, but removing any point from $S^1$ leaves it connected.

**Theorem 12.** If $X$ and $Y$ are compact, then so is $X \times Y$.

*Proof.* Next time! \qed

**Corollary 13.** By induction, this holds for any finite number of compact spaces.

**Remark.** This actually holds for even uncountably many compact spaces. This is called Tychnoff’s theorem, and is equivalent to the axiom of choice.
10/2/2019 - Compactness, Uncountability, Metric Spaces

We will begin by proving the following theorem introduced last lecture.

**Theorem.** Let $X$ and $Y$ be compact. Then $X \times Y$ is compact.

By induction we have the following corollary.

**Corollary.** Let $X_1, \ldots, X_n$ be compact. Then $X_1 \times \ldots X_n$ is compact.

**Proof.** Let $\mathcal{A}$ be an open cover of $X \times Y$. We want to find a finite subcover of $\mathcal{A}$. A basis element of $X \times Y$ is of the form $U \times V$, where $U \subset X$ and $V \subset Y$ is open. Thus each element of $\mathcal{A}$ is the union of such subsets $U \times V$.

The strategy will be to define a new cover that consists of only basis elements and demonstrate that these have a finite subcover. Then by replacing each basis element $U \times V$ with the open set of $\mathcal{A}$ in which it is contained, we obtain a finite subcover of $\mathcal{A}$. This reduces the problem to finding a subcover of a cover that consists of only basis elements, so we can assume that all sets in $\mathcal{A}$ are of the form $U_i \times V_i$, with $U_i \subset X$ and $V_i \subset Y$ open.

Consider a point $x \in X$. $\{x\} \times Y$ is homeomorphic to $Y$ and hence compact. Then it has a finite subcover of the form $\bigcup_{i=1}^n U_i \times V_i$ and $x \in U_i$ for all $i$. If we take the finite intersection $W = \bigcap_{i=1}^n U_i$, then $W$ is an open neighborhood of $x$ in $X$. Also, $\bigcup_{i=1}^n U_i \times V_i$ is a finite cover of $W \times Y$.

For every $x$, similarly define an open set $W_x$ to obtain a strip $W_x \times Y$ and a finite subcover of this strip. The sets $W_x$ for all $x$ cover $X$, so by compactness of $X$ there is a finite subcover $W_{x_1}, \ldots, W_{x_m}$. Finitely many sets from $\mathcal{A}$ cover each $W_{x_i} \times Y$, so collecting these all together yields a finite subcover of $\mathcal{A}$ for all of $X \times Y$. \qed

Recall that we proved compact Hausdorff spaces have some nice properties. Namely, if $X$ is Hausdorff and $A \subset X$ is compact, then $A$ is closed. Also, if $f : X \to Y$ is a continuous bijection between compact, Hausdorff spaces $X$ and $Y$ then $f$ is a homeomorphism. We have the following neat application of these ideas.
Uncountability of $\mathbb{R}$

We first introduce the following definition.

**Definition.** Let $X$ be a topological space. An isolated point of $X$ is a point $x \in X$ such that the singleton $\{x\}$ is open.

**Theorem.** If $X$ is a nonempty, compact Hausdorff space with no isolated points, then $X$ is uncountable.\(^a\)

We first need the following lemma.

**Lemma.** If $U \subset X$ is open and $x \in X$, there exists a nonempty open set $V$ with $x \not\in \overline{V}$ and $V \subset U$.

**Proof.** Choose $y \in U$ such that $x \neq y$. This is possible because $U$ is a neighborhood of $x$ and $x$ is not an isolated point. By Hausdorffness, there are disjoint neighborhoods $W_x$ of $x$ and $W_y$ of $y$. Take $V = W_y \cap U$, which is nonempty as it contains $y$. Then $W_x$ is open and disjoint from $V$, so $\overline{V} \subset X \setminus W_x$ and $x \not\in \overline{V}$.

We now prove the theorem.

**Proof.** Let $f: \mathbb{N} \to X$ be any function. We will show that $f$ is not a surjection. This will imply that $X$ is not countable.\(^b\) We will define a sequence of sets by induction. By the claim, set $U = X$ and find $V_1 \subset X$ such that $f(1) \not\in \overline{V_1}$. For $n > 1$, apply the claim to the point $f(n)$ and $U = V_{n-1}$. Then $\overline{V_1} \supset \overline{V_2} \supset \ldots$ is a sequence of nonempty, closed sets with $f(n) \not\in \overline{V_n}$.

We claim $\bigcup_i (X \setminus \overline{V_i}) \neq X$. Suppose for contradiction that we have equality. Then since $X$ is compact, there is a finite subcover $X \setminus V_{i_1}, \ldots, X \setminus V_{i_n}$. But any point in $\overline{V_j}$ with $j$ larger than $i_1, \ldots, i_n$ is not covered by these sets, so this is not actually a cover. Thus $\bigcap_i \overline{V_i} \neq \emptyset$.

If we take $x \in \bigcap_i \overline{V_i}$, then by definition $x \neq f(n)$ for any $n$, and $f$ is not surjective.

**Corollary.** Every closed interval of $\mathbb{R}$ is uncountable.

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**Compactness in metric spaces**

Recall that $A \subset \mathbb{R}^n$ is compact if $A$ is closed in bounded.\(^{22}\) This agrees with the topological definition.

**Theorem.** $A \subset \mathbb{R}^n$ is compact if and only if $A$ is closed and bounded in the Euclidean metric.

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\(^a\) This is a special case of a general result in topology and functional analysis called the Baire category theorem.  

**Challenge:** look up the Baire category theorem on Wikipedia and show that our result follows as a corollary.  

\(^b\) A countable set is one that admits a surjection from $\mathbb{N}$ by definition.  

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\(^{22}\) This is the Heine-Borel characterization of compactness in $\mathbb{R}^n$.  

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42
Proof. Suppose $A \subset \mathbb{R}^n$ is compact. Then $A$ is closed, since $\mathbb{R}^n$ is Hausdorff. Cover $A$ with the open balls $\{B_r(0) : r \in \mathbb{N}\}$. Then by compactness, there is a finite subcover $B_{r_1}(0), \ldots, B_{r_m}(0)$. Then there is some $r$ with $A \subset B_r(0)$, so $A$ is bounded.

Suppose $A \subset \mathbb{R}^n$ is closed and bounded. Since $A$ is bounded, it is contained in some suitably large rectangle $[-r, r]^n$. This closed rectangle is the product of intervals and thus compact. $A$ is a closed subspace of a compact space, so $A$ is compact. \qed

Remark. The theorem depends on the Euclidean metric in an important way. We can define other metrics on $\mathbb{R}^n$ that induce the standard topology, but for which this theorem is not true.

For example, the uniform metric on $\mathbb{R}^n$ induces the same topology as the Euclidean metric, but all of $\mathbb{R}^n$ is bounded in this metric (while $\mathbb{R}^n$ is not compact).

We can use compactness to generalize two of the most important theorems in calculus to compact spaces.

Theorem (Extreme value theorem). If $X$ is compact and $f : X \to \mathbb{R}$ is a continuous function, then $f$ achieves its maximum. Namely, there exists $c \in X$ such that $f(x) \leq f(c)$ for all $x \in X$.

Proof. $f(X) \subset \mathbb{R}$ is compact, so it is bounded and closed (and hence contains its limit points). If $m = \sup(X)$ is in $f(X)$, then we are done. Otherwise, $(m - \epsilon, m) \cap f(X) \neq \emptyset$ for all $\epsilon > 0$ by definition of the supremum, so $m$ is a limit point of $f(X)$ and therefore $m \in f(X)$. \qed

For the next theorem,\textsuperscript{23} we introduce a few definitions.

Definition. If $(X, d)$ is a metric space and $A \subset X$ is nonempty, the distance from $x \in X$ to $A$ is defined to be $d(x, A) = \inf\{d(x, y) : y \in A\}$.

If $A$ is compact, then there exists a point $y \in A$ with $d(x, y) = d(x, A)$. This is because $d(x, \cdot) : A \to \mathbb{R}$ is a continuous function from a compact set $A$ and achieves its minimum at some point $y \in A$.

Definition. If $A$ is bounded, the diameter of $A$ is defined to be $\sup\{d(x, y) : x, y \in A\}$.

Intuitively, the diameter of $A$ is the largest distance between two points in $A$. If $A$ is compact, then there exist points $x, y \in A$ with $d(x, y)$ equal to the diameter of $A$. This is because $d : A \times A \to \mathbb{R}$ is a continuous function\textsuperscript{24} from the compact set $A \times A$ to $\mathbb{R}$ and achieves its maximum at some pair $(x, y) \in A \times A$.

The following useful lemma will be essential for the proof.

\textsuperscript{23}Recall that the uniform continuity theorem says that if $f : [a, b] \to \mathbb{R}$ is continuous, then $f$ is uniformly continuous. This means that for any $\epsilon > 0$, there exists $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$. In other words, $\delta$ does not depend on the point $x$.

\textsuperscript{24}Verification: $d : A \times A \to \mathbb{R}$ is a continuous function.
**Lemma.** Let \( A \) be an open cover of a metric space \((X, d)\). If \( X \) is compact, then there exists some \( \delta > 0 \) such that all subsets of \( X \) of diameter less than \( \delta \) are contained in an element of \( A \). \( \delta \) is the **Lebesgue number** of \( A \).

**Proof.** Choose a finite subcover \( \{A_1, \ldots, A_n\} \subset A \). Define the function

\[
    f : X \to \mathbb{R} \\
    x \mapsto \frac{1}{n} \sum_{i=1}^{n} d(x, X \setminus A_i)
\]

If \( x \not\in A_i \), then \( d(x, X \setminus A_i) = 0 \). Intuitively, each summand measures how far the exterior of \( A_i \) is from the point \( x \).

\( f \) is the sum of continuous functions and is hence continuous. Since each \( A_i \) is open, if \( x \in A_i \) there is some \( \epsilon > 0 \) with \( x \in B_\epsilon(x) \subset A_i \). Then in such a case, \( d(x, X \setminus A_i) > \epsilon \). Any \( x \in X \) is contained in some \( A_i \), so \( f(x) > 0 \) for all \( x \in X \).

\( X \) is compact, so \( f \) achieves its minimum \( \delta > 0 \) with \( f(x) \geq \delta \) for all \( x \in X \). Then for any \( x \), there exists some \( A_i \) such that \( d(x, X \setminus A_i) \geq \delta \) by definition of \( f \) (as \( f \) is the average of all \( d(x, X \setminus A_i) \)). This \( \delta \) is the Lebesgue number of \( A \).

We now confirm the result. Suppose \( B \) has diameter less than \( \delta \). If \( x_0 \in B \) then

\[
x_0 \in B \subset B_\delta(x_0) \subset A_i
\]

\( \Box \)

We now define uniform continuity.

**Definition.** Let \((X, d_X)\) and \((Y, d_Y)\) be metric spaces. A function \( f : X \to Y \) is **uniformly continuous** if for all \( \epsilon > 0 \), there exists \( \delta > 0 \) such that \( d_X(x, y) < \delta \) implies \( d_Y(f(x), f(y)) < \epsilon \).

We will prove the following theorem next lecture.

**Theorem** (Uniform continuity theorem). If \( X \) and \( Y \) are metric spaces and \( X \) is compact, then any continuous function \( f : X \to Y \) is uniformly continuous.
Recall:  
- **X is compact** if every open cover \( X = \bigcup_{i \in I} U_i \) has a finite subcover \( X = U_i, U_i \ldots U_i \).
- \( \mathbb{R}^n \), compact \( = \) closed and bounded w.r.t. usual distance 
  (Not true in general, but \( \Rightarrow \) in metric spaces).
- \( f: X \rightarrow Y \) continuous, \( X \) compact \( \Rightarrow f(\cdot) \) compact.
  (special case: extreme value then for real-valued functions).
- **Lebesgue number lemma:**
  \( (X,d) \) compact metric space, \( A \) open cover of \( X \Rightarrow \exists \, S > 0 \) st.
  any subset of diameter \( < S \) is entirely contained in a single open of \( A \).

**Proof:** (last time).  Idea: by compactness, can assume \( A \) is a finite cover \( A_i \).

The function \( d(x, X-A_i) \) is continuous, so achieves its min, which is therefore \( > 0 \) \( \forall x \in X \exists i \) st. \( x \in A_i \) and \( d(z, X-A_i) > 0 \).

This is the magic of compactness!

counterexamples:
- \( \mathbb{R} \) is union of intervals with overlaps of lengths \( \rightarrow 0 \)
- \( \mathbb{R}^2 \) is \( U, \) \( U_2 \) centered at \( n \), length \( 1 + \frac{1}{n} \).

**Uniform continuity:**

**Definition:** \( f: (X,d_X) \rightarrow (Y,d_Y) \) is uniformly continuous if

\[ \forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x, x_1 \in X, \quad d_X(x, x_1) < \delta \Rightarrow d_Y(f(x), f(x_1)) < \varepsilon. \]

(compare with continuity: the same \( \delta \) must work for every \( x_0 \) !)

**Theorem:** If \( X \) and \( Y \) are metric spaces, \( f: X \rightarrow Y \) continuous, and \( X \) is compact, then \( f \) is uniformly continuous.

**Proof:** take \( \varepsilon > 0 \), and consider open cover of \( Y \) by balls of radius \( \frac{\varepsilon}{2} \)

so if \( f(x_0), f(x_1) \) land in same ball, they're less than \( \varepsilon \) apart.

\( X = \bigcup f^{-1}(B_{\frac{\varepsilon}{2}}(y)) \) open cover, so by Lebesgue number lemma \( \exists \, S > 0 \) st. \( \forall y \in Y \)

if \( d_X(x_0, x_1) < S \) then they lie in the same element of the cover, hence \( d_Y(f(x_0), f(x_1)) < \varepsilon \).
Limit point compactness:

We've defined compactness in terms of open covers. You may have heard of a different characterization of compactness in metric spaces in terms of convergent subsequences. These aren't equivalent to each other in general. Let's find out more...

**Def:** A space $X$ is **limit point compact** if every infinite subset of $X$ has a limit point.

**Ex:**
1) $(0,1]$ isn't limit point compact: $\{ \frac{1}{n} \mid n \in \mathbb{Z}_+ \}$ has no limit point. $\mathbb{R}$ isn't either; $\mathbb{Z}$ has no limit point.
2) $\{ \frac{1}{n} \mid n \in \mathbb{Z}_+ \} \cup \{0\}$ is limit point compact: every infinite subset has 0 as limit point.

**Thm:** $X$ is compact $\implies X$ is limit point compact.

**Pr:** Assume $X$ is not limit point compact. So $X$ contains some infinite subset $A$ with no limit point. Since $A$ contains all of its limit points (there are none), $A$ is closed. Moreover, for any $a \in A$, $a$ is a limit point so $\exists U_a \ni a$ neighborhood of $a$ s.t. $U_a \cap A = \{a\}$. The open cover $X = \left( \bigcup_{a \in A} U_a \right) \cup (X - A)$ has no finite subcover, since each $a \in A$ only belongs to $U_a$ and not to any other element of the cover. Hence $X$ isn't compact.

For a general top space, limit point compactness doesn't imply compactness.

**Ex:** give $\mathbb{Z}$ the topology generated by the basis consisting of $\{-n,n\} \cup \{n\} \forall n \in \mathbb{Z}_+$ and $\emptyset$ for any infinite subset (in fact any non-empty subset in this case!) $S \subset \mathbb{Z}$, let $n \in S - \{0\}$, then $-n$ is a limit point since every neighborhood of $-n$ contains $n \in S$, $n \neq -n$. So this is limit point compact. However it isn't compact, since the open cover by the basis open has no finite subcover.

For metric spaces, the two notions are equivalent - we'll see this soon. But first, let's introduce yet another notion: **sequential compactness**

**Def:** $X$ is **sequentially compact** if every sequence of points in $X$ has a convergent subsequence.

**Ex:** in $\mathbb{R}$, $0,0,0,0,1,\ldots$ has $1,1,\ldots$ no convergent subsequence. $1,1,2,1,2,\frac{1}{3},3,\frac{1}{2},\frac{1}{4},\ldots$ has $1,\frac{1}{2},\frac{1}{3},\ldots$ no convergent subseq. but $1,2,3,4,\ldots$ has no convergent subseq, so $\mathbb{R}$ isn't seq. compact!

In spaces with countable bases of neighborhoods, eg. metric spaces, using the relation between limit points of $ACX$ and limits of sequences of points of $A$, this ends up being equivalent to limit point compactness. In general we only have sequential compactness $\implies$ limit point compactness.
In general, sequential compactness ≠ compactness... but

**Theorem.** If \((X,d)\) is a metric space (w/ metric topology), then

\[ X \text{ is compact } \iff X \text{ is limit point compact } \iff X \text{ is sequentially compact.} \]

**Proof.**

- compact \(\implies\) limit point compact: already done (for all top spaces).

- limit point compact \(\implies\) sequentially compact: suppose \(X\) metric space and limit point compact, and consider a sequence \(x_1, x_2, \ldots\) in \(X\). If \(\{x_1, x_2, \ldots\}\) finite, then \(\exists x \in X\) s.t. \(x_n \to x\) for infinitely many \(n\), which gives a subsequence that converges to \(x\).

Otherwise, \(\{x_1, x_2, \ldots\}\) is infinite, so has a limit point \(a\).

Since every ball around \(a\) contains only many elements of the set \(\{x_1, x_2, \ldots\}\), first choose \(n_1 \in \mathbb{N}\) s.t. \(x_{n_1} \in B_{1/2}(a)\), then inductively, given \(n_1 < \cdots < n_{k-1}\), choose \(n_k > n_{k-1}\) s.t. \(x_{n_k} \in B_{1/k}(a)\). Then \(x_{n_1}, x_{n_2}, \ldots\) is a subsequence that converges to \(a\). (\(d(x_{n_k}, a) < \frac{1}{k}\) so \(\to 0\)).

- seq compact \(\implies\) compact: this is the hardest part, & requires insight. First we show:

**Lemma 1.** If \(X\) metric space is seq compact, then \(\forall \varepsilon > 0\) \(X\) can be covered by finitely many open balls of radius \(\varepsilon\).

(as we expect if \(X\) is to be compact — the cover \(X = \bigcup_{x \in X} B_{\varepsilon}(x)\) should have a finite subcover!)

**Proof:** assume not, and choose \(x_1 \in X\), then inductively choose \(x_n \in X \setminus \bigcup_{i=1}^{n-1} B_{\varepsilon}(x_i)\) (if this isn't possible then we've covered \(X\) by finitely many balls).

This yields a sequence in \(X\), which by sequential compactness must have a convergent subsequence. But this is impossible since no two terms of the sequence are within \(\varepsilon\) of each other! So the process must stop, i.e. finitely many balls suffice to cover \(X\). \(\square\)

**Lemma 2.** If \(X\) metric space is sequentially compact then every open cover has a Lebesgue number (\(\exists \delta > 0\) s.t. any subset of diameter \(< \delta\) is entirely in one element of the cover).

(we've seen this hold for compact metric space, so it should hold!)

**Proof:** suppose \(X\) open cover has no Lebesgue number, i.e. \(\forall \varepsilon \in \mathbb{R}_+ \exists C_\varepsilon \subseteq X\) with diameter \(< \frac{1}{n}\) which isn't contained in any element of the cover. Take \(x_n \in C_n\).

By sequential compactness, \(\exists\) subsequence \((x_{n_k})\) of \((x_n)\) that converges to some \(a \in\) .... 
Now we can prove \textit{seq. compact} $\Rightarrow$ \textit{compact}:

If given an open cover $X = \bigcup_{i \in \mathbb{N}} U_i$, by lemma 2 \exists \ S > 0 \ st \ every \ subset \ of \ diameter < S \ is \ entirely \ inside \ a \ single \ U_i$. Fix $\epsilon \in (0, \frac{S}{2})$: by lemma 1, $X$ is covered by finitely many $\epsilon$-balls. Each of these has diameter $\leq 2\epsilon < S$, so is contained in some $U_i$. This gives a finite subcover, replacing each $\epsilon$-ball by one $U_i$ containing it (and discarding the rest of the $U_i$'s). \qed
Compactifications:

Ex: \( \mathbb{R}^n \) is not compact, but \( \mathbb{R}^n \cup \{ \infty \} \) with basis given by open balls and
\[ U_r = \{ x \in \mathbb{R}^n \mid |x| > r \} \cup \{ \infty \} \] for \( r > 0 \) is compact. - see homework.
This is called a compactification of \( \mathbb{R}^n \).

Def: IF \( Y \) is compact (Hausdorff) and \( X \subseteq Y \) is an embedding (i.e. homeomorphism onto its image, so can view \( X \) as a subspace of \( Y \)) such that \( X \) is dense in \( Y \), then \( Y \) is called a compactification of \( X \). IF \( Y - X \) is a single point, \( Y \) is called a one-point compactification of \( X \).

Ex: 1) \( S^1 \) is a compactification of \( (0,1) \), but so is \([0,1]\) - compactification are not necessarily unique!

2) \((0,1)\times(0,1)\) has many different compactifications, for instance
\[ \begin{align*}
\text{or} & & \text{or torus} & & \text{or } S^2 \text{ (add a single point)} \\
[0,1] \times [0,1] & & & & S^2 (\text{add a single point})
\end{align*} \]

3) projective space \( \mathbb{RP}^n \) (resp \( \mathbb{CP}^n \)) is a compactification of \( \mathbb{R}^n \) (resp \( \mathbb{C}^n \))

4) \( \mathbb{Z} \) with discrete topology, let \( X = \mathbb{Z} \cup \{ \infty \} \), given subspace topology in \( \mathbb{R} \cup \{ \infty \} \)
looks like
\[ \begin{array}{c}
\cdots -2 & -1 & 0 & 1 & 2 & \cdots \\
\end{array} \]
This gives a one-point compactification of \( \mathbb{Z} \).

Compactifications are really useful, e.g. in algebraic geometry compact varieties are easier to work with.
When do they exist & have good properties? An important notion in this context is:

Local compactness:

Def: \( X \) is locally compact at \( x \) if there exists a compact subset \( C \subseteq X \) which contains a neighborhood of \( x \). \( X \) is locally compact if it is locally compact at every point.

Ex: • any compact space is locally compact
• \( \mathbb{R}^n \) is locally compact: \( x \in B_r(x) \subseteq \overline{B_r(x)} \) compact since closed & bounded
\[ \mathbb{R}^n \text{ with product topology is not locally compact, since none of its basis elements are contained in compact subspaces (otherwise their closures would be compact).} \]

9. Local compactness at \( 0 \) would require some \( 0 \in \left\{ (-\varepsilon, \varepsilon) \right\} \cap \cdots \cap \left\{ (-\varepsilon, \varepsilon) \right\} \subset R \subset R \subset \cdots \subset \mathbb{R} \text{ compact} \]

& hence \( \left[ -\varepsilon, \varepsilon \right] \times \cdots \times \left[ -\varepsilon, \varepsilon \right] \times \cdots \text{ would be compact.} \]

Taking open cover by \( U_n = \left\{ x = (x_1, x_2, \ldots) \mid x_{n+1} \in (-n, n) \right\} \), this has no finite subcover.

Local compactness turns out to be exactly the necessary & sufficient condition for a Hausdorff space \( X \) to have a Hausdorff one-point compactification.

\[ \text{Thm:} \quad X \text{ is locally compact Hausdorff } \iff \exists Y \text{ s.t. } \]

(1) \( X \) is a subspace of \( Y \)
(2) \( Y - X \) is a single point
(3) \( Y \) is compact Hausdorff.

\[ \text{Moreover, such } Y \text{ is unique up to homeomorphism.} \]

\[ \text{Prof:} \quad \iff \text{(loc. compactness is necessary): assume } Y = X \cup \{ \infty \} \text{ compact Hausdorff.} \]

\[ \text{X is a subspace of } Y \text{ Hausdorff, so it's Hausdorff.} \]

If \( x \in X \), choose \( U \ni x \) and \( V \ni \infty \) disjoint neighborhoods in \( Y \); let \( C = Y - V \).

\[ x \in U \subset C = Y - V \subset Y - \{ \infty \} = X. \]

\[ \text{So } X \text{ is locally compact at } x. \]

\[ \text{nbd. of } x \text{ closed in } Y, \text{ hence compact.} \]

\[ \iff \text{define as a set, } Y = X \cup \{ \infty \} \quad \text{one new element (} \infty \neq X, \text{ else find some other symbol for the new point).} \]

\[ \text{& define topology on } Y, \]

\[ T_Y = \{ U \cup U \cap C \text{ open} \} \cup \{ Y - C \mid C \subset X \text{ compact} \} \]

\[ \text{types: the open } \not\in \infty \text{ are exactly those of } X \]

\[ \text{the open } \in \infty \text{ have complement } \text{ a compact subset of } X \text{.} \]

\[ \text{Step 1: this is a topology.} \]

- \( \phi \text{ type 1, } Y = Y - \phi \text{ type 2.} \]

- \( \{ \text{arbitrary unions } \} \{ \text{finite intersections } \} \text{ of type 1 opens are type 1 opens} \)

- \( \{ \text{type 1} \} \text{ opens are} \quad \{ \text{type 2} \} \text{ open if } U \cap C \text{ open, } C \subset X \text{ compact (hence closed) then } \\
  \quad \text{closed in } C \text{ hence compact} \)

- \( \text{if } U \cap C \text{ open, } C \subset X \text{ compact (hence closed) then } \\
  \quad \text{closed in } C \text{ hence compact} \)

- \( \text{Also, since } X \text{ is open in } T_Y \text{ (type 1), subspace topology induced by } T_Y \text{ on } X \text{ is} \\
  \quad \text{all type 1 opens, } \text{i.e. the topology } T_X \text{ we started with.} \)
Step 2. \( Y \) is Hausdorff; can separate points of \( X \) by type 1 opens; to separate \( x \in X \) from \( \infty \), local compactness \( \Rightarrow \exists C \supseteq U \exists x, \) U and \( Y-C \) separate \( x \) and \( \infty \).

Step 3: \( Y \) is compact; if \( Y = \bigcup A_i \) open cover, \( \infty \) is in some \( A_i \in Y-C \) compact.

and now \( U(A_i \cap C) \) form an open cover of \( C \) which is compact.

so \( \exists i_1 \ldots i_n \) in \( C = A_i \cup u \ldots u A_i \). Hence \( Y = A_i \cup (A_i \cup u \ldots u A_i) \) finite subsets. so \( Y \) is compact.

Uniqueness up to homeomorphism:

assume \( Y' = X \cup \{p\} \) compact Hausdorff & subspace topology on \( X \) coincide with \( T_x \).

We show the opens in \( Y' \) must be exactly as provided above & the only difference with \( Y \) is the naming of the added point, so the map \( Y \to Y' \) defined by \( x \to x \) gives a homeomorphism between \( Y \) and \( Y' \).

\( \{p\} \) is closed since \( Y' \) Hausdorff, hence \( X \) is open in \( Y' \).

So the subspace topology on \( X \) consists exactly of open subsets of \( Y' \) which \( \subseteq X \). So the type 1 opens in \( Y' \) are exactly the open sets of \( X \).

Type 2 opens; if \( V \subseteq Y \) is open and \( p \in V \), then \( C = Y' - V \) is closed in \( Y \) compact, hence compact. But in fact \( \subset X \) is compact. Conversely, if \( C \subseteq X \) is compact then it is closed in \( Y' \) (\( Y' \) Hausdorff) and so \( Y' - C \) must be open in \( Y' \).

The definition we've given for local compactness doesn't look very "local" ('local" usually means something that can be checked inside an arbitrarily small neighborhood of a given point). Here's a better re-formulation, if \( X \) is Hausdorff.

**Prop.** Assume \( X \) is Hausdorff. then \( X \) is locally compact \( \iff \forall x \in X, \forall U \exists x \) neighborhood such a neighborhood \( U \) of \( x \) s.t. \( \overline{V} \subseteq U \) and \( V \) is compact.

**Proof:** \( \Rightarrow \) take \( U = X \ni x, \) then \( \exists V \supseteq V \ni x, \) compact open which is the definition of local compactness at \( x \).

\( \Leftarrow \) let \( x \in X, \) \( U \ni x \) neighborhood. let \( Y \) be one-point compactification of \( X \).

\( Y \) is compact Hausdorff, and \( C := Y - U \) is closed in \( Y \) hence compact.

Lemmas: in a Hausdorff space, if \( C \) is compact & \( x \) disjoint from \( C \) then \( \exists \) disjoint open subsets \( V \ni x, V \ni C \). (i.e. in a Hausdorff space, we can separate points from compact subsets.)
Given this, \( x \in V \subseteq \overline{V} \subseteq y - V' \subseteq y - C = \emptyset \).

\[ \text{open} \quad \Rightarrow \quad V' \text{ open so } y - V' \text{ closed,} \]
\[ \text{closed in } \overline{y} \text{ hence compact} \quad y - V' \supseteq V \Rightarrow y - V' \supseteq \overline{V} \]

**Proof of Lemma:**

\( x \notin C \) so \( \forall y \in C, \exists V_y \ni x, V_y \ni y \) disjoint open.

\( C \) compact \( \Rightarrow \) \( C = \bigcup_{y \in C} V_y \) has a finite subcover, \( C \subseteq V' = V_{y_1} \ldots V_{y_n} \).

Then \( V = V_{y_1} \cap \ldots \cap V_{y_n} \) open \( \exists x \), and disjoint from all \( V_y \); hence disjoint from \( V' \).

Next time (=next Wed) we'll see more instances of separation properties –
we'll focus on normal space & metrizability.
We return to structural properties of top spaces (countability & separation axioms) with a goal of better understanding (next time) which topologies are induced by metrics.

**Countability axioms:** (§30)

**Def:** A space has a **countable basis** of neighborhoods at \( x \in X \) if there exists a countable collection \( \{ U_n \}_{n \in \mathbb{Z}_+} \) of neighborhoods of \( x \), such that every neighborhood \( V \ni x \) contains at least one of the \( U_n \).

A space that has a countable basis of nbhd at each point is said to be **first-countable**.

**Ex:** metric space are first-countable: at \( x \in X \), take \( U_n = B_{1/n}(x) \).

(in general, without loss of generality, by intersecting we can assume \( U_1 \supset U_2 \supset U_3 \supset \ldots \))

We've mentioned before that this is the property which makes limits of sequences relate accurately to limit points:

Thus: a \( X \) top space, \( x \in X \), if some sequence \( x_n \in X \) converges to \( x \) then \( x \in \overline{A} \). If \( X \) is first-countable then, conversely, if \( x \in \overline{A} \) then \( \exists \) sequence \( x_n \in A \), \( x_n \to x \).

A stronger requirement is:

**Def:** If the topology on \( X \) is generated by a countable basis, then \( X \) is said to be **second-countable**.

**Ex:** \( \mathbb{R} \) has a countable basis: open intervals \( (a,b) \) s.t. \( a,b \in \mathbb{Q} \).

\( \mathbb{R}^n \) has a countable basis (boxes whose corners have rational coordinates) and even \( \mathbb{R}^\omega \) with the product topology has countable basis (same idea: products of intervals which are \( (a,b) \), \( a,b \in \mathbb{R} \) in finitely many coordinates, \( \mathbb{R} \) in all others. This is still countable).

**Ex:** \( \mathbb{R}^\omega \) with uniform topology doesn't have countable basis

(even though it is first-countable, since topology come from a metric)

Indeed, \( \{0,1\}^\omega \subset \mathbb{R}^\omega \) uncountable subset, discrete in uniform top.

so every basis of \( (\mathbb{R}^\omega, \text{uniform}) \) must contain uncountably many basis elements each containing only one point of \( \{0,1\}^\omega \).
Prop: If $X$ has countable basis then $X$ has a dense subset which is countable.

(Proof: Pick one point in each non-empty basis open. The resulting subset $A$ is countable, and intersects every open hence $\overline{A} = X$.)

Ex: $\mathbb{Q}$ has a countable dense subset ($\mathbb{Q}$) but doesn't have a countable basis (see HW 2).

Regular and normal spaces (531-32)

Recall: $X$ Hausdorff $\iff$ can separate points: $\forall x \neq y, \exists U \ni x, \exists V \ni y$ disjoint open (aka $T_2$) $\iff T_1$: $\{x\}$ is closed $\forall x \in X$.

Stronger separation axioms:

Def: Suppose one-point subsets $\{x\} \subset X$ are closed ($T_1$). Then say
- $X$ is regular if $\forall x \in X, \forall B \ni x$ closed disjoint from $x$, $\exists$ disjoint open sets $U \ni x, V \ni B$.
- $X$ is normal if $\forall A, B \subset X$ disjoint closed subsets, $\exists$ disjoint open sets $U \ni A, V \ni B$.

Normal ($T_4$) $\Rightarrow$ Regular ($T_3$) $\Rightarrow$ Hausdorff ($T_2$) $\Rightarrow$ $T_1$

Ex: $\mathbb{R}^2$ is normal. Indeed: Given $A, B$ disjoint closed:

$\forall a \in A$, $\mathbb{R}^2 - B$ open so $\exists \varepsilon_a$ st: $[a, a + \varepsilon_a)$ disjoint from $B$.

$\forall b \in B$, $\exists \varepsilon_b$ st: $[b, b + \varepsilon_b)$ disjoint from $A$.

Take $U = \bigcup_{a \in A} [a, a + \varepsilon_a)$ open $\supset A$, $V = \bigcup_{b \in B} [b, b + \varepsilon_b)$ open $\supset B$.

Can show $U \cap V = \emptyset$. (Roughly: "because we've only extended $A$ & $B$ to the right")

(Usual $\mathbb{R}$ is normal too but need to be a bit more careful when continuing $U \cap V$. See below.)

- $\mathbb{R}^2$ with the product topology is regular but not normal! (hard. see 531 Ex. 3)
- $\mathbb{R}^n$ are normal. $\mathbb{R}^n$ with product or uniform topology is normal.

If $J$ is uncountable then $\mathbb{R}^J$ with product top. is regular but not normal.
Every regular space w/ countable basis is normal.

(see Munkres Thm 32.1)

Every metric space is normal.

Proof: Let $A, B$ disjoint closed $\subset (X, d)$. $\forall a \in A$, $\exists \varepsilon_a > 0$ s.t. $B_{\varepsilon_a}(a) \cap X-B$. $\forall b \in B$, $\exists \varepsilon_b > 0$ s.t. $B_{\varepsilon_b}(b) \cap X-A$.

Define $U = \bigcup_{a \in A} B(a, \varepsilon_a/2)$ and $V = \bigcup_{b \in B} B(b, \varepsilon_b/2)$.

Why not just $\varepsilon_a$ & $\varepsilon_b$?

$U \supset A, V \supset B$ are open (unions of open balls).

Claim $U \cap V = \emptyset$. Indeed: if $x \in U \cap V$ then $\exists a \in A, b \in B$ s.t. $x \in B(a, \varepsilon_a/2) \cap B(b, \varepsilon_b/2)$. Hence $d(a, b) \leq d(a, x) + d(x, b)$

$$< \frac{\varepsilon_a + \varepsilon_b}{2} \leq \max(\varepsilon_a, \varepsilon_b).$$

This is a contradiction (e.g. if $d(a, b) < \varepsilon_a$ then $B(a, \varepsilon_a)$ isn't disjoint from $B$ as claimed!).

Every compact Hausdorff space is normal.

Proof: We showed last time: in a Hausdorff space, we can separate points from compact subsets. This implies a compact Hausdorff space is regular.

Recall: $x \in X$, $B \subset X$ closed (hence compact) with $x \notin B$.

$\forall y \in B$, $\exists U_y \ni x$, $U_y \cap B$ disjoint open.

$B$ compact $\Rightarrow \exists y_1, \ldots, y_n \in B$ s.t. $V = U_{y_1} \cup \cdots \cup U_{y_n} \supset B$.

Then $U = U_{y_n} \杯 \cdots \cup U_{y_1}$ is a neighborhood of $x$, disjoint from $V$.

To prove $X$ is normal, we essentially repeat the same argument. $A, B \subset X$ disjoint closed (hence compact) subsets $\Rightarrow$

$\forall x \in A$, $\exists U_x \ni x$, $U_x \supset B$ disjoint open.

$A$ is compact so $\exists x_1, \ldots, x_n \in A$ s.t. $V = U_{x_1} \cup \cdots \cup U_{x_n} \supset A$.

And then $V = \cup U_{x_n} \cap \cdots \cap U_{x_1} \supset B$ open, $U \cap V = \emptyset$.

We've seen that metrizable $\Rightarrow$ first-countable and normal. There are various metrization theorems which give partial converses.
We'll focus on one that is particularly simple to state (but still requires some clever ideas in order to prove!); the Urysohn metrization theorem.

**Theorem:** If $X$ is regular + has a countable basis, then it is metrizable.

The first condition is necessary, but the second one is stronger than necessary.

**Nagata-Smirnov theorem** gives a necessary & sufficient condition:

$X$ is metrizable iff $X$ is regular + has a "countably locally finite basis".

(we won't discuss this further. Countably locally finite basis means: $\mathcal{B}$ is the union of countably many subsets $B_1, B_2, \ldots$, each of which is locally finite i.e. $V \in X \ni B \subseteq X$ and intersecting only finitely many of $B_i$.)
Urysohn metrization theorem: If $X$ is regular and has a countable basis, then it is metrizable.

(Recall: regular $\iff$ can separate points from closed sets. However, regular + countable basis $\Rightarrow$ normal, i.e. can separate closed sets from each other.)

The key ingredient in this theorem is Urysohn's lemma:

**Theorem:** A normal space, $A, B$ disjoint closed subsets

$\Rightarrow \exists$ continuous $f : X \to [0,1]$ s.t. $f(x) = 0 \forall x \in A$

$f(x) = 1 \forall x \in B$.

**Idea:** 1) construct open sets $U_q$, $\forall q \in [0,1] \cap \mathbb{Q}$ s.t. $A \subset U_0 \subset \ldots \subset U_1 = X - B$

and in fact $p < q \Rightarrow \overline{U_p} \subset U_q$. (Do this using $X$ normal).

2) set $U_q = X$ for $q > 1$.

Step 1: Care the following reformulation of normality:

**Lemma:** $X$ is normal $\iff \forall A \text{ closed, } \forall U \supseteq A \text{ open, } \exists \text{ open } V \text{ s.t. } A \subset V \text{ and } \overline{V} \subset U$.

**Proof:** $A$ and $B = X - U$ are disjoint closed sets, so since $X$ is normal,

$\exists V \supseteq A$, $V' \supseteq B$ open such that $V \cap V' = \emptyset$.

Moreover, $X - V'$ closed, $V \subset X - V' \Rightarrow \overline{V} \subset X - V'$.

So $A \subset V \subset \overline{V} \subset X - V' \subset X - B = U$. \hfill \square

**Proof of Urysohn's lemma:**

**Step 1:** Given $A \& B$ disjoint closed, let $U_0 = X - B$, and let $U_0$ open s.t. $A \subset U_0 \subset \overline{U_0} \subset U_1$.

Next, we construct $U_q$, $q \in (0,1) \cap \mathbb{Q}$, s.t. $p < q \Rightarrow \overline{U_p} \subset U_q$.

Do this by induction, choosing a labelling of $[0,1] \cap \mathbb{Q} = \{q_0, q_1, q_2, q_3, \ldots\}$

by an infinite sequence such that $q_0 = 0$ & $q_1 = 1$.

(could eg continue: $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \ldots$).

Assuming $U_{q_0}, \ldots, U_{q_n}$ have already been chosen, we construct $U_{q_{n+1}}$ using the above lemma:

namely let $q_{n+1} = \max \left\{ q_0, q_1, \ldots, q_n \right\} \cap [0, q_{n+1})$ so $q_k < q_{n+1} < q_l$ & none of the rationals already considered lie in between.
Then by induction hypothesis, \( U_{q_k} \subseteq U_{q_k} \), hence using normality:

\[ \exists \text{ open } V \text{ s.t. } U_{q_k} \subseteq V \subseteq U_{q_\alpha}, \text{ let } U_{q_{\alpha + 1}} = V. \]

By induction, we construct in this way all the \( U_{q} \)'s.

and indeed \( p < q \Rightarrow U_{p} \subseteq U_{q} \).

We also set \( U_{q} = \emptyset \) if \( q < 0 \)
\[ U_{q} = X \text{ if } q > 1. \]

(* trivial case: \( p < q \Rightarrow U_{p} \subseteq U_{q} \). *)

**Step 2:** Define \( f(x) = \inf Q_x \), where \( Q_x = \{ q \in \mathbb{Q} / x \in U_q \} \).

Notes:
- \( f(x) \leq 1 \) \( \forall x \in X \), since \( x \in U_q \) \( \forall q > 1 \), so \( \inf \leq 1 \).
- if \( x \in B \) then \( x \notin U_q \Leftrightarrow x \in U_q \subseteq X - B \) so \( Q_x = \mathbb{Q} \cap (1, \infty) \) and \( f(x) = 1 \).
- \( f(x) \geq 0 \) \( \forall x \in X \), since \( Q_x = [0, \infty) \) \( (U_q = \emptyset \text{ for } q < 0) \).
- if \( x \in A \subseteq U_0 \) then \( 0 \notin Q_x \) and \( f(x) = 0 \).

So it only remains to show that \( f : X \to [0, 1] \) is continuous! For this, observe:
- \( x \in U_q \Rightarrow f(x) \leq q \), indeed if \( x \in U_q \) then \( x \notin U_{q'} \) \( \forall q' > q \), so \( Q_x \supseteq \mathbb{Q} \cap (q, \infty) \).
- \( x \notin U_q \Rightarrow f(x) \geq q \), indeed if \( x \notin U_q \) then \( Q_x \subseteq \mathbb{Q} \cap (q, \infty) \).

Now, we can prove \( f^{-1}((c, d)) \) is open in \( X \) for open interval \( (c, d) \).

Assume \( x_0 \in f^{-1}((c, d)) \), and let \( p, q \in \mathbb{Q} \) s.t. \( c < p < f(x_0) < q < d \).

Then by the above observation, \( x_0 \in U_q \) and \( x_0 \notin U_p \).

\[ V = U_q \cap (X - U_p) \] is open, and a neighborhood of \( x_0 \).

Moreover, \( x \in V \Rightarrow x \notin U_p \) so \( f(x) \geq p \) hence \( x \in V \subseteq f^{-1}([p, q]) \).

Hence \( x_0 \in V \subseteq f^{-1}((p, q)) \).

This proves \( f^{-1}((c, d)) \) is open (contains a neighborhood of each of its points).

& so \( f \) is continuous.

Now we prove the metrization theorem, namely that if \( X \) is normal & has countable basis, then \( X \) is metrizable. We actually do this by embedding \( X \) as a subspace of a metric space, namely \([0, 1]^\kappa\) with product topology or uniform topology — in fact both come from metrics.

Indeed, uniform metric on \([0, 1]^\kappa\) \( d_\infty((x_n), (y_n)) = \sup_n |x_n - y_n| \)

Product top: \( d'_\infty((x_n), (y_n)) = \sup_n |x_n - y_n| \) then \( B_{\epsilon}(k_n) = \Pi (x_n - \epsilon, x_n + \epsilon) \)
Step 1: There exists a countable collection of continuous functions $f_n : X \to [0,1]$ s.t. $\forall x \in X$, $\forall U \ni x$, there exists a neighborhood $V$ s.t. $f_n(x) > 0$ and $f_n = 0$ on $X - U$.

Proof: This follows from Urysohn's lemma, but need to be careful so that countably many functions suffice.

Let $B = \{B_n\}$ be a countable basis for $X$. If $x \in U$, then $\exists B_n \in B$ s.t. $x \in B_n \subseteq U$.

But then, since $X$ is normal, $\exists V$ open s.t. $x \in V \subseteq \overline{V} \subseteq B_n$, and $\exists B_m \in B$ s.t. $x \in B_m \subseteq V$, so that $x \in \overline{B_m} \subseteq B_n \subseteq U$.

So, for each $(m,n) \in \mathbb{Z}_+ \times \mathbb{Z}_+$ s.t. $\overline{B_m} \subseteq B_n$, apply Urysohn's lemma to get

$g_{m,n} : X \to [0,1]$ s.t. $g_{m,n} = 1$ on $B_m$ and $0$ on $X - B_n$.

This countable collection of functions has the stated property.

Step 2: $F : X \to [0,1]^\mathbb{N}$, product topology is an embedding.

$x \mapsto F(x) = (f_1(x), f_2(x), \ldots)$

(hence $x$ is homeo to $F(x) \subseteq [0,1]^\mathbb{N}$, so topology on $X$ is defined by the metric $d_F = \|F(x) - F(y)\|_{\infty}$).

Proof: $F$ is continuous in product topology because each component $f_1, f_2, \ldots$ is continuous $\forall x \in [0,1]$.

- $F$ is injective, since $x \neq y \Rightarrow \exists U \ni x$, $V \ni y$ disjoint open
  $\Rightarrow \exists m, n$ s.t. $f_m(x) > 0$, $f_n = 0$ outside of $U$ (hence $d_F$)
  $f_m(y) > 0, f_n = 0$ outside of $V$ (hence $d_F$).
- Finally, must show that $F$ is a homeo $X \rightarrow \mathbb{Z} = F(X) \subseteq [0,1]^\mathbb{N}$.
  Since $F$ is a continuous bijection $X \rightarrow \mathbb{Z}$, only remains to prove:
  if $U \subseteq X$ open then $F(U) \subseteq \mathbb{Z}$ is open.

For this, let $U \subseteq X$ be any open set, and $x_0 \in U$. Then $\exists n$ s.t. $f_n(x) > 0$ and $f_n = 0$ outside of $U$. Let $V_n = \pi^{-1}(0, \infty) \cap \mathbb{N}$ = \{\:(z_1, z_2, \ldots) \in \mathbb{Z} \mid z_n > 0\} \subseteq \mathbb{N}

Then $x_0 \in F^{-1}(V_n) \subseteq U$ (since $f_n(x) > 0$, and $f_n(x) > 0 \Rightarrow x \in U$).

Hence $F(x_0) \in V_n \subseteq F(U)$, this is true $\forall x_0 \in U$ ($\Rightarrow$ $F(x_0) \in F(U)$)

Open in $\mathbb{Z}$, so we conclude that $F(U)$ is open.

Hence $F : X \rightarrow \mathbb{Z}$ is a homeo, and we conclude $X$ is homeo to a metric space.

Remark: without the assumption that $X$ has a countable basis, this still produces embeddings into some $[0,1]^\mathbb{N}$ with product topology — however this isn't retrievable for $X$ uncountable.
Categories

**Def.** A category is a collection of objects and for each pair of objects, a collection of morphisms \( \text{Mor}(A, B) \), and an operation called composition of morphisms,
\[
\text{Mor}(A, B) \times \text{Mor}(B, C) \rightarrow \text{Mor}(A, C) \quad \text{st.}
\]
\[
f, g \quad \mapsto \quad g \circ f
\]

1) every object \( A \) has an identity morphism \( \text{id}_A \in \text{Mor}(A, A) \) st. \( \forall f \in \text{Mor}(A, B), \quad f \circ \text{id}_A = \text{id}_B \circ f. \)

2) composition is associative: \( (f \circ g) \circ h = f \circ (g \circ h) \).

Check: \( \text{id}_A \) is unique (if \( e_A \) is an identity morphism then \( e_A = e_A \circ \text{id}_A = \text{id}_A \)).

**Ex.**
1) category of sets, \( \text{Mor}(A, B) = \) all maps \( A \rightarrow B \)
2) finite dim. vector spaces/\( \mathbb{R}^n \), linear maps.
3) groups, group homomorphisms (function homomorphisms)
4) top. spaces, continuous maps.

**Def.** \( f \in \text{Mor}(A, B) \) is an **isomorphism** if \( \exists g \in \text{Mor}(B, A) \) st. \( g \circ f = \text{id}_A \) and \( f \circ g = \text{id}_B \) (the inverse isomorphism)

Check: the inverse of \( f \), if it exists, is unique.
- \( \text{id}_A \) is an isomorphism; \( f \) iso \( \Rightarrow f^{-1} \) iso; \( f, g \) isos \( \Rightarrow g \circ f \) isos.

The automorphisms of \( A \), \( \text{Aut}(A) = \{ \text{isomorphisms } A \rightarrow A \} \subset \text{Mor}(A, A) \), form a group.

**Ex.**
1) In Sets, A finite set with \( n \) elements \( \Rightarrow \text{Aut}(A) = \{ \text{bijective } A \rightarrow A \} \cong \mathbb{S}_n \)
2) \( V = \text{n-dim. vector space/\( \mathbb{R}^n \) } \Rightarrow \text{Aut}(V) \cong \mathbb{G}(n, \mathbb{R}) \) (invertible \( n \times n \) matrices).

**Claim:** If \( A \cong B \) then \( \text{Aut}(A) \cong \text{Aut}(B) \) isomorphic as groups.
(The isomorphism depends on the choice of iso \( f \in \text{Mor}(A, B) \)).

**Claim/Exercise:** A group is the same thing as a category with a single object where all morphisms are isomorphisms.
(given group \( G \), consider cat. with a single object \( * \), \( \text{Mor}(\cdot, \cdot) = G \), composition = product in \( G \))
**Def.** A groupoid is a category in which all morphisms are isomorphisms.

The difference with a group is we can't always multiply two elements, need the objects to match. However, when composition is defined it does have the same properties as multiplication in a group: associativity \((f \circ g) \circ h = f \circ (g \circ h)\)

identity \((f \circ \text{id}_A = \text{id}_B \circ f = f)\)

inverse \((f^{-1} \circ f = \text{id}_B, f \circ f^{-1} = \text{id}_A)\).

**Examples:**
- sets w/ bijections
- groups w/ isomorphisms
- top spaces w/ homeomorphisms.

**Functors:** in alg. topology, we will associate to a top. space \(X\) an algebraic object \(A(X)\) (a set, a group, ...). We want the assignment of \(X \mapsto A(X)\) to be a functor from the category of top. spaces to another category, so that \(Vf, X \rightarrow Y\) continuous induces a morphism \(A(f): A(X) \rightarrow A(Y)\).

**Def.** \(C, D\) categories. A functor \(F: C \rightarrow D\) is an assignment
- to each object \(X\) in \(C\), an object \(F(X)\) in \(D\).
- to each morphism \(f \in \text{Mor}_C(X, Y)\), a morphism \(F(f) \in \text{Mor}_D(F(X), F(Y))\)

st. 1) \(F(id_X) = id_{F(X)}\)

2) \(F(g \circ f) = F(g) \circ F(f)\).

**Ex:**
1) **Forgetful functor** taking a group, a top. space, ... to the underlying set.
2) on vector spaces, given a vect. space \(V\), \(F: W \rightarrow \text{Hom}(V, W)\)
   if \(f: W \rightarrow W\) is linear, then induced map \(\text{Hom}(V, W) \overset{F(f)}{\rightarrow} \text{Hom}(V, W)\)
   \(a \mapsto f \circ a\).
3) **Sets \rightarrow Groups**
   \(X \mapsto \text{free group generated by } X\). Eg. \(F\{a, b\} = \langle a, b \rangle\) free group on two generators = set of finite words in \(a, b, a^{-1}, b^{-1}\), where multiplication is concatenation (\& simplification \(a^{-1}a = 1\), ...).
   eg. \((aba) \cdot (a^{-3}ba) = aba^{-2}ba\).

\((F\{a\}) = \{a^n | n \in \mathbb{Z}\} \subseteq \mathbb{Z}\).
Homotopy = notion of continuous deformation, parameterized by \( I = [0,1] \).

**Def:** \( f, g : X \to Y \) two continuous maps. A homotopy between \( f \) and \( g \) is a continuous map \( F : I \times I \to Y \) st. \( F(x,0) = f(x) \quad \forall x \in X \), \( F(x,1) = g(x) \).

(\( I \) the "time" variable in the homotopy)

If this exists, then say \( f \) and \( g \) are homotopic and write \( f \simeq g \).

If \( f \) is homotopic to a constant map, we say it is nullhomotopic.

We'll want to study paths in top spaces, i.e. \( f : [0,1] \to X \) continuous, \( f(0) = x_0, f(1) = x_1 \).

The above notion of homotopy is not very useful for paths if we don't fix the end points \( x_0, x_1 \) (see Thm). A better notion of homotopy of paths only considers homotopies which keep the end points in place.

\[ f \simeq_p g \text{ homotopic paths} \]

\[ h \text{ not homotopic to } f \& g \]

**Def:** Two paths \( f, g : I \to X \) from \( x_0 \) to \( x_1 \) are (path) homotopic if there exists continuous \( F : I \times I \to X \) st. \( F(s,0) = f(s), F(s,1) = g(s) \) (homotopy) and \( F(0,t) = x_0, F(1,t) = x_1 \) (fix end points; so \( \forall t \in [0,1], \quad F(t) = x \) is a path from \( x_0 \) to \( x_1 \)).

Such \( F \) is a path homotopy, and we write \( f \simeq_p g \).

**Lemma:** \( \simeq \) and \( \simeq_p \) are equivalence relations.

**Pf:**
- clearly \( f \simeq f \) (constant homotopy \( F(x,t) = f(x) \)).
- if \( f \simeq g \) with homotopy \( F(x,t) \), then the reverse homotopy \( G(x,t) = F(x,1-t) \) gives \( g \simeq f \).
- Assume \( f \simeq g \) with homotopy \( F(x,t) \) then the concatenation of these is \( g \simeq h \quad \text{ and } \quad G(x,t) \)

\[ H : X \times [0,1] \to Y \text{ defined by } H(x,t) = \begin{cases} F(x,2t) & \text{if } t \in [0,\frac{1}{2}] \\ G(x,2t-1) & \text{if } t \in [\frac{1}{2}, 1] \end{cases} \]

These two formulas agree at \( t = \frac{1}{2} \) \( G(x,1) = g(x) = G(x,0) \) so \( H \) is well-defined and continuous (cf. "pasting lemma" Thm 18.3), and gives a homotopy \( f \simeq h \).

- In the case of paths homotopies, can check the above construction preserves the requirements \( F(0,t) = x_0 \) & \( F(1,t) = x_1 \), so yield path homotopies.

We denote the (path) homotopy equivalence class of \( f \) by \([f]\).
Ex. 1) If \( f, g \) are paths in \( \mathbb{R}^2 \) (or \( \mathbb{R}^n \)) from \( x_0 \) to \( x_1 \), we can define the **straight-line homotopy**

\[
F(s,t) = (1-t)f(s) + tg(s)
\]

For each \( s \), this connects \( f(s) \) to \( g(s) \) by a straight line segment. We conclude, \( f \sim_p g \) always!

2) In the punctured plane \( X = \mathbb{R}^2 - \{(0,0)\} \), let \( f, g \) be paths from \((-1,0)\) to \((0,0)\) such that \( f \) stays in the upper half plane \( \{(x,y) \mid y \geq 0\} \) and \( g \) stays in the lower half plane \( \{(x,y) \mid y \leq 0\} \).

Then there is no homotopy between \( f \) and \( g \) in \( X \).

(We'll prove this rigorously later.)

---

Now we prove that homotopy classes of paths form a category, in fact a groupoid.

The key operation is composition (concatenation) of paths:

**Def.** If \( f \) is a path from \( x \) to \( y \) and \( g \) is a path from \( y \) to \( z \), define a path \( f \circ g \) from \( x \) to \( z \) by running through \( f \) first if \( f \) then \( g \) (twice as fast):

\[
(f \circ g)(s) = \begin{cases} f(2s) & \text{if } s \in [0, \frac{1}{2}] \\ g(2(s-1)) & \text{if } s \in [\frac{1}{2}, 1] \end{cases}
\]

This product is well-defined on path-homotopy classes, as long as \( f(1) = g(0) \): if \( f \sim f' \) and \( g \sim g' \) then \( f \circ g \sim f' \circ g' \).

Using homotopy \( (F \circ G)(s,t) = \begin{cases} F(2s,t) & \text{if } s \leq \frac{1}{2} \\ G(2(s-1),t) & \text{if } s \geq \frac{1}{2} \end{cases} \)

So we can define

\[
[f] \circ [g] = [f \circ g]
\]

(Still assuming end points match, \( f(1) = g(0) \)).

Claim: this operation is associative, and has identity & inverse,

so that path-homotopy classes form a groupoid with objects = points of \( X \) and \( \text{Mor}(x,y) = \{ \text{homotopy classes of paths from } x \text{ to } y \} \).

(empty unless \( x \) and \( y \) are in the same path-connected component of \( X \)!!)

- **Identity:** given \( x \in X \), consider the constant path \( e_x : I \to X \), \( e_x(s) = x \) \( \forall s \), & let \( id_x = [e_x] \).
- **Inverse:** given a path \( f \) from \( x \) to \( y \), define the reverse path \( \overline{f}(s) = f(1-s) \) from \( y \) to \( x \).
Recall, we've discussed path homotopy and composition (concatenation) of paths in a top space.

**Def:** if \( f \) is a path from \( x \) to \( y \) and \( g \) is a path from \( y \) to \( z \), define a path \( f \circ g \) from \( x \) to \( z \) by running through first \( f \) then \( g \) (twice as fast):

\[
(f \circ g)(s) = \begin{cases} f(2s) & s \in [0, \frac{1}{2}] \\ g(2s-1) & s \in [\frac{1}{2}, 1] \end{cases}
\]

This product is well-defined on path-homotopy classes, as long as \( f(1) = g(0) \):

If \( f \sim_p f' \) and \( g \sim_p g' \), then \( f \circ g \sim_p f' \circ g' \).

So we define \([f] \circ [g] = [f \circ g]\)

**Claim:** this operation is associative, and has identity & inverses, so that path-homotopy classes form a **groupoid** with objects = points of \( X \) and \( \text{Mor}(x,y) = \{ \text{homotopy classes of paths from } x \text{ to } y \} \).

(Sometimes called the "fundamental groupoid" of \( X \).)

**Identity:** given \( x \in X \), consider the constant path \( e_x : I \to X \), \( e_x(s) = x \) \( \forall s \), & let \( \text{id}_x = [e_x] \).

We claim that if \( f \) is any path from \( x \) to \( y \), then \([f] \circ \text{id}_y = \text{id}_x \circ [f] = [f]\).

Indeed, there are explicit homotopies:

\[
F(s,t) = \begin{cases} f(\frac{s}{1-t/2}) & s \in [0, 1-\frac{t}{2}] \\ y & s \in [\frac{1}{2}, 1] \end{cases}
\]

and similarly \([e_x \circ f] = [e_y \circ f] \), \( F(s,t) = \begin{cases} f(\frac{s-t/2}{1-t/2}) & s \in [\frac{1}{2}, 1]) \\ x & s \in [0, \frac{1}{2}] \end{cases} \)

(same contraction after reversing \( s \)-direction).

**Inverse:** given a path \( f \) from \( x \) to \( y \), define the reverse path \( \overline{f}(s) = f(1-s) \) from \( y \) to \( x \).

\([f] \) is inverse to \([f] \), namely \( e_x \sim_p f \circ \overline{f} \) and \( e_y \sim_p \overline{f} \circ f \). Indeed:

\[
F(s,t) = \begin{cases} f(2ts) & s \in [0, 1]] \\ f(2t(1-s)) & s \in [\frac{1}{2}, 1] \end{cases}
\]

For given \( t \), this runs forward along \( f \) from \( f(0) = x \) to \( f(t) = x \) at \( s = \frac{1}{2} \) then backwards to \( f(0) = x \) at \( s = 1 \). For \( t = 0 \) get \( e_x \) (similarly for \( e_y \)).

(Similarly for \( e_y \)).
- Associativity: given paths \( f, g, h \) with \( f(0) = g(0) = h(0) \), claim \((f * g) * h \simeq f * (g * h)\). Both run along \( f \) then \( g \) then \( h \), but with different parameterizations. The homotopy comes from adjusting for this:

\[
(f * (g * h))
\]

\[
\text{Let } F(s,t) = \begin{cases} 
  f\left(\frac{4s}{1+t}\right) & s \in [0, \frac{1+t}{4}] \\
  g\left(4s - (1+t)\right) & s \in \left[\frac{1+t}{4}, \frac{2+t}{4}\right] \\
  h\left(\frac{4s - (2+t)}{2-t}\right) & s \in \left[\frac{2+t}{4}, 1\right]
\end{cases}
\]

Fundamental group:

- The inability to multiply every pair of paths (because the groupoid has more than one object) prevents us from having a group structure. To address this, we usually restrict to a single object of the category, i.e. we fix a base point \( x_0 \in X \) and only consider paths which go from \( x_0 \) to itself — i.e. loops (based at \( x_0 \)).

**Def.** The set of path homotopy classes of loops based at \( x_0 \), with operation \( \ast \) (concatenation), is called the fundamental group of \( X \), denoted \( \pi_1(X, x_0) \).

**Ex:** in \( \mathbb{R}^n \) (or a convex domain in \( \mathbb{R}^n \)), every loop at \( x_0 \) is path-homotopic to the identity (i.e. the constant path at \( x_0 \)) by the straight line homotopy:

\[
F(f, s) = (1-t)f(s) + tx_0
\]

So \( \pi_1(\mathbb{R}^n, x_0) = \{ \text{id} \} \).

**Def.** \( X \) is simply-connected if \( X \) is non-empty, path-connected, and for \( x_0 \in X \),

\[
\pi_1(X, x_0) = \{ 1 \}.
\]

**Ex:** \( \mathbb{R}^n \), convex subspace of \( \mathbb{R}^n \), the n-sphere \( S^n \) for \( n \geq 2 \).

Dependence on the base point:

If \( x_0, x_1 \) are in the same path-connected component of \( X \), let \( \alpha \) be a path from \( x_0 \) to \( x_1 \).

Then for any loop \( f \) based at \( x_0 \), we get a loop at \( x_1 \) by taking \( \overline{\alpha} \ast f \ast \alpha \),

and so we get a map \( \hat{F} : \pi_1(X, x_0) \rightarrow \pi_1(X, x_1) \)

\[
[F] \mapsto [\overline{\alpha} \ast f \ast \alpha] = [\alpha]^{-1} *[f] * [\alpha]
\]

(recall \( \ast \) well-defined on path-homotopy classes).
Prop. \( \pi_1(X,x_0) \rightarrow \pi_1(X,x_0) \) is a group isomorphism.

Formally, this follows from: \( x_0 \) and \( x_1 \) an isomorphic in the fundamental groupoid, with \([α]\) giving an isomorphism, so \( \text{Aut}(x_0) \cong \text{Aut}(x_1) \) as groups. Let's do it explicitly.

**Proof.** If \( a, b \in \pi_1(X,x_0) \) then \( \tilde{α}(a \cdot b) = [α]^{-1} \cdot (a \cdot b) \cdot [α] \)

using associativity & inverse: \( = [α]^{-1} \cdot a \cdot [α] \cdot b \cdot [α] \)

So \( \tilde{α} \) is a group homomorphism.

Let \( β = \tilde{α}^{-1} \) reverse path from \( x_1 \) to \( x_0 \), then \( \tilde{β} : \pi_1(X,x_1) \rightarrow \pi_1(X,x_0) \).

We claim \( \tilde{β} \) and \( \tilde{α} \) are inverses of each other. Indeed: for \( a \in \pi_1(X,x_0) \),

\[ \tilde{β}(\tilde{α}(a)) = \tilde{β}(α^{-1} \cdot a \cdot α) = [β]^{-1} \cdot [α]^{-1} \cdot a \cdot [α] \cdot [β] \]

(recall \( [β] = [α]^{-1} \) in groupoid) \( = [α] \cdot [α]^{-1} \cdot a \cdot [α] \cdot [α]^{-1} = a \).

Hence \( \tilde{β} \circ \tilde{α} = \text{id} \) (and similarly \( \tilde{α} \circ \tilde{β} = \text{id} \) as well), so \( \tilde{α} \) is an isomorphism.

**Corollary 1.** If \( X \) is path-connected, then \( \pi_1(X,x_0) \) is independent of \( x_0 \) up to isomorphism.

**Corollary 2.** A loop \( f \) at \( x_0 \) induces an automorphism \( f \) of \( \pi_1(X,x_0) \).

This group automorphism is of the form \( a \rightarrow [f]^{-1} \cdot a \cdot [f] \),

called an inner automorphism. ("cajugation" by \([f]\))

**π₁ as a functor.** Consider the category of pointed topological spaces:

- **Objects:** top. space + choice of base point, \((X,x_0)\)
- **Morphisms:** continuous maps preserving base points: \( f : (X,x_0) \rightarrow (Y,y_0) \) means \( f : X \rightarrow Y \) continuos & s.t. \( f(x_0) = y_0 \).

**Def/Prop.** A continuous map \( h : (X,x_0) \rightarrow (Y,y_0) \) induces a group homomorphism \( h_* : \pi_1(X,x_0) \rightarrow \pi_1(Y,y_0) \) defined by \( h_*([f]) = [h \circ f] \).

**Check:** if \( f \sim f' \) via \( F \) then \( h \circ f \sim h \circ f' \) via \( h \circ F \). So \( h_* \) is well-defined.

\( h \circ (f \cdot g) = (h \circ f) \cdot (h \circ g) \) (composition w/ \( h \) compatible with concatenation).

So \( h_* \) is a group homomorphism, \( h_*([f] \cdot [g]) = h_*([f]) \cdot h_*([g]) \).
Claim: \( \pi_1 \) is a functor from pointed top. spaces to groups.

(a objects, \( \pi_1(X,x_0) \); on morphisms, \( \pi_1(h) = h_*: \pi_1(X,x_0) \to \pi_1(Y,y_0) \))

Proof:
- composition: \( h: (X,x_0) \to (Y,y_0) \) and \( k: (Y,y_0) \to (Z,z_0) \) continuous

  \( \Rightarrow \) if \( [f] \in \pi_1(X,x_0) \), then \( (k \circ h)_*[f] = ([k \circ h] \circ [f]) = [k \circ (h \circ f)] \)

  \( = k_*([h \circ f]) = k_*([h] \circ [f]) \).

- identity: \( (id_X)_*[f] = [id_X \circ f] = [f] = id([f]). \)

Since functors map isomorphisms to isomorphisms, we get:

Conclusion: if \( h: (X,x_0) \to (Y,y_0) \) is a homeomorphism, then \( h_* \) is an isomorphism.

(In fact there are much more general statements, once we get to the notion of homotopy equivalence of top. spaces.)
Last time: \((X, x_0)\) pointed top space \(\mapsto\) fundamental group \(\pi_1(X, x_0) = \{\text{homotopy classes of loops in } X \text{ based at } x_0\}\), product = concatenation (id = constant loop, inverse = backward).

At some point we'd like to show \(\pi_1(S^1) \cong \mathbb{Z}\). We'll do this by introducing a key tool for the study of \(\pi_1\): the notion of covering spaces.

**Def.** Let \(p: E \to B\) be a continuous surjective map. We say \(p\) **evenly covers** an open subset \(U \subset B\) if \(p^{-1}(U) = \bigsqcup_{a \in A} V_a\), where \(V_a \subset E\) are disjoint open subsets, and for each \(a \in A\), \(p|_{V_a} : V_a \to U\) is a homeomorphism. The \(V_a\) are called **slices**.

(equivalently: \(\exists p^{-1}(U) \overset{\text{homeo}}{\longrightarrow} U \times A\) denote the restriction map. \(p|_{U} = p \times_{\text{id}} \tau\).

**Def.** If every point of \(B\) has a neighborhood which is evenly covered by \(p\), we say \(E\) is a covering space of \(B\) and \(p\) is a covering map.

**Ex.** Define \(p: \mathbb{R} \to S^1\)

\[
p(t) = (\cos t, \sin t)
\]

This is a covering map! For instance consider \((1, 0) \in S^1\) and the neighborhood \(U = \{(x, y) \in \mathbb{S}^1 / x > 0\}\).

Then \(p^{-1}(U) = \bigsqcup_{n \in \mathbb{Z}} (2\pi n - \frac{\pi}{2}, 2\pi n + \frac{\pi}{2})\) and \(p\) is a homeomorphism on each slice.

Thus: \(p: E \to B\), \(q: E' \to B'\) covering maps \(\Rightarrow p \circ q: E \times E' \to B \times B'\) is a covering map.

**Thm.** Given \((b, b') \in B \times B'\), let \(U \supset b\) and \(U' \supset b'\) be neighborhoods.

\[
p^{-1}(U) = \bigsqcup V_a, \quad q^{-1}(U') = \bigsqcup V_b \text{ slice, then}
\]

\[
(p \circ q)^{-1}(U \times U') = \bigsqcup_{a, b} V_a \times V_b \text{ union of open slices homeo to } U \times U'.
\]

**Ex.** Consider the torus \(S^1 \times S^1\): since \(\mathbb{R}\) covers \(S^1\), \(\mathbb{R}^2\) covers \(S^1 \times S^1\)
**Ex.** if X any top space, A set w/ discrete topology, then \( p_1 : X \times A \to X \) is a covering map.

**Exercise (on HW):** if \( B \) connected \& \( p : E \to B \) covering map, then for any two points \( x, y \in B \), \( p'(x) \) and \( p'(y) \) have the same cardinality. 
If \( \#p'(x) = d < \infty \) then \( d \) is called the degree of the covering.

**Ex:** consider \( S^1 = \{ z \in \mathbb{C} \mid |z| = 1 \} \), then \( p : S^1 \to S^1 \) (so \( e^{i\theta} \mapsto e^{i\theta} \)) is an n-fold covering.

---

**Lifting:**

**Def.** Given \( p : E \to B \) continuous map, a \underline{lifting} of \underline{a continuous map} \( f : X \to B \) is a map \( \tilde{f} : X \to E \) s.t. \( p \circ \tilde{f} = f \).

\[ \xrightarrow{\tilde{f}} \text{ Lifts } \]

If \( p : E \to B \) is a covering map, then we can \underline{locally lift}, namely if \( f(x) \in U \subseteq B \) and \( U \) is evenly covered, then we can lift \( f \) to one of the sheets.

We'll see that if \( p : E \to B \) is a covering then paths and path homotopies in \( B \) can always be lifted.
Ex: consider \( p: \mathbb{R} \to S^1 \) and the path \( f(s) = (\cos \pi s, \sin \pi s): I \to S^1 \) with the diagram:

\[
\begin{array}{c}
-2\pi & \cdots & -\pi & 0 & \pi & 2\pi & 3\pi & \cdots \\
\sim f & \nearrow & \downarrow P & f & \searrow & \cdots
\end{array}
\]

This has infinitely many possible lifts to paths in \( \mathbb{R} \), depending on where \( 0 \) gets lifted to.

**Theorem:** \( p: E \to B \) covering map, \( f: [0,1] \to B \) a path starting at \( f(0) = b \), and 
\( e \in p^{-1}(b) \). Then there exists a unique lift \( \widetilde{f}: [0,1] \to E \) s.t. \( \widetilde{f}(0) = e \).

**Proof:**

Cover \( B \) by open sets \( U_\alpha \) which are evenly covered by \( p \). Then the preimages \( p^{-1}(U_\alpha) \) are an open cover of \([0,1]\), which is compact, so \( \exists \) Lebesgue number \( \delta > 0 \) s.t. \( (x, x+\delta) \subset p^{-1}(U_\alpha) \) for some \( x \). Hence we can find a finite subdivision:

\[ 0 = s_0 < s_1 < \ldots < s_n = 1 \]

such that each \( f([s_i, s_{i+1}]) \) lies inside one of the \( U_\alpha \).

Define \( \widetilde{f}(s) = e \). Assume we have defined \( \widetilde{f}(s) \) for \( s \in [0,s_i] \). Then we define \( \widetilde{f}(s) \) for \( s \in [s_i, s_{i+1}] \) as follows. Recall \( f([s_i, s_{i+1}]) \subset U \) for some \( U \) which is evenly covered by \( p \), \( p^{-1}(U) = \bigcup \) slices. Let \( V \) be the slice which contains \( \widetilde{f}(s) \). The map \( p_{|V}: V \to U \) is a homeomorphism, so has a continuous inverse \( \widetilde{p}^{-1}(f(s)) \) for \( s \in [s_i, s_{i+1}] \), which extends \( \widetilde{f} \) continuously over \([s_i, s_{i+1}]\). Repeating the process, we obtain a continuous lift \( \widetilde{f}: [0,1] \to E \).

\( \widetilde{f} \) is unique since for each \( s \), there was a unique slice containing \( \widetilde{f}(s) \) and a unique way to lift \( f([s_i, s_{i+1}]) \) into it.

**Theorem:** Let \( F: I \times I \to B \) be continuous with \( F(0,0) = b \), \( p: E \to B \) a covering map, 
\( e \in p^{-1}(b) \), then \( \exists \) unique lift \( \widetilde{F}: I \times I \to E \) s.t. \( \widetilde{F}(0,0) = e \).

The proof is exactly the same, subdividing \( I \times I \) into squares of side length \( \delta \) which map into open subsets of \( B \) that are evenly covered, then continuing the lift \( \widetilde{F} \) one square at a time.

**Observe:** if \( F \) is a path-homotopy from \( f \) to \( g \) (in \( B \)), then \( \widetilde{F} \) is a path-homotopy (in \( E \)) from \( \widetilde{f} \) to \( \widetilde{g} \). Indeed, if \( F(s,t) = b \) for all \( t \), then \( \widetilde{F}(s,t) \in p^{-1}(b) \) which is a discrete subset of \( E \) (one point in each slice), so we must have \( \widetilde{F}(s,t) = e \) for all \( t \) (always the same point). Similarly for the other end point \( \widetilde{F}(s,1) \).
On the other hand, loops don’t always lift to loops!

Ex: \[ \begin{array}{ccc}
I & \hookrightarrow & \mathbb{R} \\
\downarrow f & & \uparrow p \\
S^1 & \hookrightarrow & \mathbb{R}
\end{array} \]

But since path lifting is unique, given a starting point \( e_0 \in \tilde{p}^{-1}(b_0) \), the end point is uniquely determined. So we have a map

\[
\gamma : \pi_1(B,b_0) \longrightarrow p^{-1}(b_0)
\]

defined by \( \gamma([f]) = \tilde{f}(1) \) where \( \tilde{f} \) is the lift of \( f \) st. \( \tilde{p}(0) = e_0 \).

This is called the lift correspondence.

Q: Why is \( \gamma \) well-defined? (i.e. independent of choice \( \tilde{f}, \tilde{g} \) in its homotopy class?)

A: if \( F \) is a path homotopy \( f \simeq_p g \), then its lift \( \tilde{F} \) starting at \( e_0 \) is a path homotopy between \( \tilde{f} \) and \( \tilde{g} \), so \( \tilde{f}(1) = \tilde{g}(1) \).

Ex: for the covering \( p : \mathbb{R} \to S^1 \), taking \( b_0 = (1,0) \), \( e_0 = 0 \in \mathbb{R} \),

if \( f \) loops around the circle \( k \) times (counting clockwise) then, its lift \( \tilde{f} \) ends at

\[
\gamma([f]) = \tilde{f}(1) = 2\pi k.
\]

This gives a map \( \pi_1(S^1, (1,0)) \to 2\pi \mathbb{Z} \) (surjective).

(next time we’ll show it’s a group isomorphism).
Recall: Given a covering map \( p: E \rightarrow B \) a path \( f: I \rightarrow B \) starting at \( f(0) = b_0 \) admits a unique lift to a path \( \tilde{f}: I \rightarrow E \) starting at \( e_0 \in p^{-1}(b_0) \)

Homotopic paths have homotopic lifts (in particular, same end point \( \tilde{f}(1) \)).

\[
\begin{align*}
\text{Ex}: & \quad p: \mathbb{R} \rightarrow S^1 \\
& \quad x \mapsto (\cos 2\pi x, \sin 2\pi x) \\
& \quad f \text{ loop going once around circle}
\end{align*}
\]

* Considering lifts \( \tilde{f} \) loops based at \( b_0 \in B \), this gives the lifting correspondence

\[
\varphi: \pi_1(B, b_0) \rightarrow p^{-1}(b_0)
\]

defined by \( \varphi([f]) = \tilde{f}(1) \) where \( \tilde{f} \) is the lift of \( f \) s.t. \( \tilde{f}(0) = e_0 \).

**Ex:** for \( p: \mathbb{R} \rightarrow S^1 \), taking \( b_0 = (1,0) \) and \( e_0 = 0 \), \( \varphi \) takes loop going \( k \) times around \( S^1 \) to the integer \( k \in \mathbb{Z} = p^{-1}(b_0) \).

**Prop:** If \( E \) is path connected then \( \varphi: \pi_1(B, b_0) \rightarrow p^{-1}(b_0) \) is surjective.

If \( e \in p^{-1}(b_0) \), \( g: I \rightarrow E \) a path from \( e_0 \) to \( e \), then \( f = p \circ g: I \rightarrow B \) is a loop at \( b_0 \) where lift starting at \( e_0 \) is \( \tilde{f} = g \). So \( \varphi([f]) = e \).

Recollecting **Prop:** If \( X \) is simply connected then any two paths \( f, g \) from \( x_0 \) to \( x_1 \) are path-homotopic

\[
\text{Pf: } f \simeq_p g \text{ is a loop at } x_0, \text{ so } f \simeq_p e_{x_0} \text{ (} X \text{ simply connected). Then } f \simeq_p f \simeq_p (g \circ g) \simeq_p f \simeq_p e_{x_0} \ast g \simeq_p g. \quad \Box
\]

**Thm:** If \( p: E \rightarrow B \) is a covering and \( E \) is simply connected, then

\[
\varphi: \pi_1(B, b_0) \rightarrow p^{-1}(b_0) \text{ is a bijection.}
\]

**Pf:** By the above, \( \varphi \) is surjective. If \( \varphi([f]) = \varphi([g]) \) then \( \tilde{f}, \tilde{g} \) are paths in \( E \) starting at \( e_0 \) and ending at the same point \( e_1 \). Since \( E \) is simply connected, \( \tilde{f} \simeq_p \tilde{g} \). Hence \( p \circ \tilde{f} \simeq_p p \circ \tilde{g} \), i.e. \( p \circ \varphi = \varphi \circ p \), so \( [f] = [g] \). So \( \varphi \) is injective. \( \Box \)

**Thm:** \( \pi_1(S^1) \cong \mathbb{Z} \)

**Pf:** consider the covering map \( p: (\mathbb{R}, 0) \rightarrow (S^1, (1,0)) \), \( p(x) = (\cos 2\pi x, \sin 2\pi x) \).

Since \( \mathbb{R} \) is simply connected, by the above the lifting correspondence

\[
\varphi: \pi_1(S^1, (1,0)) \rightarrow p^{-1}((1,0)) = \mathbb{Z}
\]

is a bijection.
We just need to show it is a group homomorphism.

Let \([f_1, g_1] \in \pi_1(S^1)\) and let \(p(f_1) = n, p(g_1) = m\).

I.e. the lifts \(\tilde{f}_1\) and \(\tilde{g}_1\) starting at 0 end at \(n\) and \(m\).

Define a new path \(h = (s) = n + \tilde{g}_1(s)\); this is the lift of \(g\) starting at \(n = \tilde{f}_1(0)\). Then \(\tilde{f}_1 h\) is a well-defined path in \(\tilde{R}\) from 0 to \(n + m\), and it is the lift of \(f \circ g\) starting at 0. So \(p(f \circ g) = n + m = p(f) + p(g)\) \(\Box\)

(Can do similarly; for \(\bigcirc\), \(\pi_1(S^1 \times S^1) \cong \mathbb{Z} \times \mathbb{Z}\), using covering prop: \(\mathbb{R}^2 \to S^1 \times S^1\).

---

**Gluing & quotients** (522)

One good way to build interesting topological spaces is by "gluing" together simpler spaces.

**Ex:**

\[ [0,1] \to \bigcirc_{s^1}, \quad \square \to \bigcirc_{s^1} \]

The construction underlying this is the **quotient topology**.

**Def.**

Let \(X\) be a top space, \(A\) a set, \(f: X \to A\) a surjective map.

The **quotient topology** on \(A\) is defined by:

\( U \subseteq A \) is open \(\iff\) \(f^{-1}(U) \subseteq X\) is open.

(Exercise: check this is a topology on \(A\), in fact the finest topology for which \(f\) is continuous.)

A map \(f: X \to Y\) between topological spaces is called a **quotient map** if

- \(f\) is surjective and \(U \subseteq Y\) is open \(\iff\) \(f^{-1}(U) \subseteq X\) is open.

With the quotient topology on \(A\), \(f: X \to A\) is a quotient map.

Say \(A\) is a **quotient space** of \(X\).

**Note:** if \(f: X \to Y\) is surjective, continuous, and open, then \(f\) is a quotient map.

Indeed:

- \(U \subseteq Y\) open \(\implies\) \(f^{-1}(U) \subseteq X\) open since \(f\) continuous.

- \(f^{-1}(U) \subseteq X\) open \(\implies U = f(f^{-1}(U))\) (using \(f\) surjective) is open since \(f\) open.

The converse is not true — there exist quotient maps which aren't open. E.g. \(p_2: \mathbb{R}^2 \to \mathbb{R}\)

---

Why is this called quotient? / how does it relate to "gluing"?

Typically, one starts from an equivalence relation \(\sim\) on \(X\), and defines \(A\) to be the set of equivalence classes \(A = X/\sim\). \(f: X \to A\) sends \(x \in X\) to its equivalence class \([x] \in X/\sim = A\). There is no loss of generality — if \(f: X \to A\) is surjective then we can define an equivalence relation on \(X\) by \(x \sim y \iff f(x) = f(y)\) and then \(X/\sim = A\).
* Example: Can think of $S^1$ as $[0,1]$ with 0 glued to 1: i.e. set $0 \sim 1$ so \{0,1\} is one equivalence class (and the other are just \{x\}, $x \in (0,1)$).

The quotient map is $f: [0,1] \to S^1$, $f(x) = (\cos 2\pi x, \sin 2\pi x)$.

vs. the map $g: [0,1] \to S^1$ defined by restricting $f$ is also surjective, but it is not a quotient map: let $U = \{(x,y) \in S^1 \mid y > 0\} \cup \{(0)\}$.

$U$ is not open in $S^1$, but

$g^{-1}(U) = [0,\frac{1}{2})$ is open in $[0,1]$.

Ex: $(x_1,x_1), \ldots, (x_n,x_n)$ pointed top spaces with each $X_i \simeq S^1$.

+ let $A = \text{quotient space of } \coprod X_i$ by the equivalence relation $x_i \sim x_j ~ \forall i,j$. (glue the $X_i$ at their basepoints). This is called the **wedge** of the circle $X_1\ldots X_n$.

\[ X_1 \quad X_2 \quad X_3 \quad \longrightarrow \quad \bigcirc \quad \bigcirc \quad \bigcirc \quad \longrightarrow \quad A \]

* If $A = X/\sim$ and $f: X \to Y$ is a map st. $x \sim x' \Rightarrow f(x) = f(x')$, then as a map of sets we can define $\overline{f}: X/\sim \to Y$ by $\overline{f}([x]) = f(x)$.

Thus: If $f: X \to Y$ is a continuous map and $x \sim x' \Rightarrow f(x) = f(x')$, then equipping $X/\sim$ with the quotient topology, $\overline{f}: X/\sim \to Y$ is a continuous map.

If $p: X \to X/\sim$ the quotient map, and recall $\overline{f}([x]) = f(x)$ (indep. of $x \in [x]$).

So $\overline{f} \circ p = f$. Hence: \forall $U \subset Y$ open, $f^{-1}(U) = p^{-1}(\overline{f}^{-1}(U)) \subset X$ is open.

By definition of the quotient topology, we conclude that $\overline{f}^{-1}(U) \subset X/\sim$ is open. \[ V \subset X/\sim \text{ open } \Rightarrow p^{-1}(V) \subset X \text{ open}. \]

Ex: \(X = \mathbb{R}^n - \{0\}\), define an equivalence relation $x \sim y$ iff $x,y$ lie on the same line through the origin, i.e. $x = ky$ for some $k \in \mathbb{R}$, $k \neq 0$. This is an equivalence relation.

The quotient space is projective (n-1)-space, $\mathbb{RP}^{n-1} = X/\sim$ with quotient topology ("space of lines through 0 in $\mathbb{R}^n$")

If $Y$ is another top space, then a continuous map $\overline{f}: \mathbb{RP}^{n-1} \to Y$ is the same thing as a continuous map $f: \mathbb{R}^n - \{0\} \to Y$ st. $f(kx) = f(x)$ \forall $x \in \mathbb{R}^n - \{0\}$, \forall $x \in X$.

We'll see later: $\mathbb{RP}^1 \simeq S^1$ hence, but $\mathbb{RP}^n \neq S^n$ for $n \geq 2$. 

Ex: Various quotients of the unit square \(X = [0,1]^2\): let the edges be
\[ A = \{0\} \times [0,1], \ A', B, B' \]
1) gluing \(A\) to \(A'\) by \((0,t)\sim(1,t)\), get a cylinder

2) if instead we glue \(A\) to \(A'\) by \((0,t)\sim(1,1-t)\), we get a **Mobius band**!

3) gluing \(A\) to \(A'\) via \((0,t)\sim(1,t)\) and \(B\) to \(B'\) by \((s,0)\sim(s,1)\) give us the torus

4) gluing \((0,t)\sim(1,t)\) and \((s,0)\sim(1-s,1)\), however, give the **Klein bottle**, which cannot be embedded in \(\mathbb{R}^3\) (can draw a picture that self-intersects).

5) gluing \((0,t)\sim(1,1-t)\) and \((s,0)\sim(1-s,1)\) is tricky to visualize, but the quotient is actually homeomorphic to \(\mathbb{R}P^2\).

(Exercise: what about gluing \((0,t)\sim(t,0)\) and \((1-s,1)\sim(1,s,0)\) — what does that look like?)
We'll now explore some topological applications of \( \pi_1(S^1) = \mathbb{Z} \) — in some sense, two-dimensional analogues of the intermediate value theorem, specifically:

1) Every continuous map \( f: I \to I \) has a fixed point \( (\exists x \in I \text{ s.t. } f(x) = x) \)

*Proof:* let \( g(x) = f(x) - x \), apply intermediate value to \( g \). \( g(0) \geq 0, \ g(1) \leq 0 \).

2) If \( f: S^1 \to \mathbb{R} \) is continuous, then \( \exists x \in S^1 \text{ s.t. } f(x) = f(-x) \).

(*this was an HW4, proof considers \( g(x) = f(x) - f(-x) \) & connectedness of \( S^1 \)).

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1. **The Brouwer fixed point theorem:**

Let \( B^n \) denote the closed ball of radius 1 in \( \mathbb{R}^n \), with boundary the unit sphere \( S^{n-1} \). Recall that if \( A \subset X \), a retrac-tion \( r: X \to A \) is a continuous map s.t. \( r(a) = a \forall a \in A \).

Then: There is no retraction of \( B^2 \) onto \( S^1 \).

*Proof:* if \( r: B^2 \to S^1 \) is a retraction, then given any loop \( f \) in \( S^1 \),

\( f \) is also a loop in \( B^2 \), which is convex \( \subset \mathbb{R}^2 \), so \( \exists \) straight line homotopy \( F: I \times I \to B^2 \) from \( f \) to constant loop. Then \( r \circ F: I \times I \to S^1 \) gives a homotopy in \( S^1 \) from \( f \) to constant loop. This contradicts \( \pi_1(S^1) \neq 1 \).

(see also HW).

[with more alg. loops, similarly \( \not\exists \) retraction \( B^n \to S^{n-1} \forall n \)].

2. **Brouwer fixed point theorem:**

If \( f: B^2 \to B^2 \) is continuous, then \( \exists x \in B^2 \text{ s.t. } f(x) = x \).

[with more alg. loops, the same holds for continuous maps \( B^n \to B^n \forall n \)].

**Proof (by hand):**

assume \( f: B^2 \to B^2 \) continuous, \( f(x) \neq x \ \forall x \in B^2 \).

Then define \( h: B^2 \to S^1 \) by mapping each \( p \in B^2 \) to the point where the ray from \( f(p) \) to \( p \) hits \( \partial B^2 = S^1 \).

(Formula: \( h(p) = p + t(p-f(p)) \) where \( t \geq 0 \) s.t. \( ||h(p)||^2 = 1 \).

(can solve by quadratic formula, so \( t \) does depend continuously on \( p \).)

This gives a continuous map \( h: B^2 \to S^1 \), moreover if \( p \in S^1 \) then \( h(p) = p \), so we get a retraction \( B^2 \to S^1 \). Contradiction.

\( \square \)

We'll give a more conceptual version of this argument, after some useful lemmas...
Lemma: Let $h: S^1 \to X$ continuous, then the following are equivalent:

1. $h$ is nullhomotopic
2. $h$ extends to a continuous map $k: B^2 \to X$ such that $k|_{B^2 - S^1} = h$.
3. $h_*: \pi_1(S^1) \to \pi_1(X)$ is the trivial homomorphism.

Proof:

(1) $\Rightarrow$ (2): Let $H: S^1 \times I \to X$ be a homotopy between $h$ and a constant map.

Define a map $\Pi: S^1 \times I \to B^2$ by $\Pi(x,t) = (1-t)x$:

\[
\begin{array}{cc}
\text{cylinder} & \rightarrow \\
\text{disk}
\end{array}
\]

can check $\Pi$ is a quotient map, collapsing $S^1 \times \{1\}$ to a point (the origin),
and a homeomorphism on $S^1 \times \{0\} \to B^2 - \{0\}$.

Since $H: S^1 \times I \to X$ is constant on $S^1 \times \{1\}$, it induces a continuous map

\[
(S^1 \times I)/\sim \to X,
\]

i.e. $\exists k: B^2 \to X$ s.t. $H = k \circ \Pi$.

Moreover, $\Pi$ maps $S^1 \times \{0\}$ to $S^1 \subset B^2$, so

\[k|_{S^1} = \partial B^2 \] agrees with $H|_{S^1 \times \{0\}} = h$.

(2) $\Rightarrow$ (3): if $h \simeq k|_{S^1}$ then one can write $h = k \circ i$ where $i: S^1 \to B^2$ is the inclusion.

By functoriality of $\pi_1$, $h_* = k_* \circ i_* : \pi_1(S^1) \xrightarrow{i_*} \pi_1(B^2) \xrightarrow{k_*} \pi_1(X)$
but $\pi_1(B^2) = \{1\}$, so $k_*$ is trivial and so is $h_*$.

(3) $\Rightarrow$ (1): assume $h_*: \pi_1(S^1, b_0) \to \pi_1(X, x_0)$ is trivial.

Let $f: I \to S^1$ be the loop $f(s) = (\cos(2\pi s), \sin(2\pi s))$, representing the generator of $\pi_1(S^1)$.

This is also a quotient map $- [0,1]/0 \sim 1 \simeq S^1$.

Then $g = h \circ f: I \to X$ is a loop in $X$, representing $h_*(f(1)) = 1$.

Thus $\exists$ path homotopy $G: I \times I \to X$ from $g$ to the constant path at $x_0$.

Now, $F: I \times I \to S^1 \times I$ is a quotient map identifying $(0,t) \sim (1,t)$,
\[(s,t) \mapsto (f(s),t) \]
and since $G(0,t) = G(1,t) = x_0 \forall t$, $G: I \times I \to X$ induce a continuous map

$H: S^1 \times I \to X$ s.t. $H \circ F = G$.

& $H$ is now a homotopy between $H|_{S^1 \times 0} = h$ and $H|_{S^1 \times 1} = \text{constant map at } x_0$.\[\square\]
The Borsuk-Ulam Theorem for $S^2$: ($\S 57$)

If $f: S^2 \to \mathbb{R}^2$ is continuous, then there exists a pair of antipodal points $x, -x \in S^2$ such that $f(x) = f(-x)$.

(similarly for $f: S^n \to \mathbb{R}^n$ in general; case $n=1$ follows from intermediate value theorem, case $n \geq 3$ requires more algebraic topology...)

Next time!
The Borsuk-Ulam Theorem for $S^2$: §57

Let $f: S^2 \to \mathbb{R}^2$ be continuous, then there exists a pair of antipodal points $x, -x \in S^2$ such that $f(x) = f(-x)$.

(Similarly for $f: S^n \to \mathbb{R}^n$ in general; case $n=1$ follows from intermediate value theorem, case $n \geq 3$ requires more algebraic topology...).

Let's first start with simple steps.

**Def.**
- If $x \in S^n$, its antipode is $-x \in S^n$. A map $h: S^n \to S^m$ is antipode-preserving if it maps antipodes to antipodes: $h(-x) = -h(x)$ for all $x \in S^n$.

**Ex.** Rotation of $S^1$ by angle $\theta$ is antipode-preserving: viewing $S^1$ as unit complex numbers, $r_\theta(z) = e^{i\theta}z$, and $r_\theta(-z) = -e^{i\theta}z = -r_\theta(z)$.

Then, if $h: S^1 \to S^1$ is continuous and antipode-preserving then it's not nullhomotopic.

**Pf.** (By hand — see Lecture for a proof that generalizes to higher dim’s)

We show $h_\#: \pi_1(S^1) \to \pi_1(S^1)$ (i.e. $\mathbb{Z} \to \mathbb{Z}$) is a nontrivial homomorphism.

Let $\alpha: S^1 \to S^1$ be the antipodal map, and let $f: I \to S^1$ be $x \mapsto -x$.

Path $b_0(1,0) \sim -b_0$ going halfway around $S^1$, so $g = f \circ (\alpha \circ f)$ is a loop going once around.

Then $h \circ f: I \to S^1$ is a path from $h(b_0)$ to $h(-b_0) = -h(b_0)$, and $h \circ \alpha = \alpha \circ h \Rightarrow h \circ \alpha \circ f = \alpha \circ (h \circ f)$, path from $-h(b_0)$ to $h(b_0)$.

The composition $(h \circ f) \times (\alpha \circ (h \circ f)) = h \circ (f \times (\alpha \circ f)) = h \circ g$ is then a loop representing the homotopy class $h_\#([g])$, which we wish to show is nontrivial.

Now, consider path-lifting to the covering space $p: \mathbb{R} \to S^1$, where lifting map gives $t \mapsto (\cos \pi t, \sin \pi t) \in \mathbb{R}^2$, $\pi_1(S^1) = \mathbb{Z}$.
Ride a lift to of $h(b_0)$, the lift $k$ of $h \circ f$ starting at $t_0$ is a path in $R$ which ends at a point of $p_1(-h(b_0)) = t_0 + \frac{1}{2} + \mathbb{Z}$, say $t_0 + n + \frac{1}{2}$ for some $n \in \mathbb{Z}$.

The lift of $x_0(h(b_0))$ starting at $t_0 + n + \frac{1}{2}$ is then $l(s) \rightarrow k(s) + n + \frac{1}{2}$, which is a path from $t_0 + n + \frac{1}{2}$ to $t_0 + 2n + 1$.

Hence the lift of $h \circ g = (h \circ f) \circ (x_0 \circ h \circ f)$ starting at $t_0$ is $h \circ k \circ l$, which is a path in $R$ from $t_0$ to $t_0 + 2n + 1$. We conclude that under $\pi_1(S^1) = \mathbb{Z}$,

$[h \circ g] \rightarrow 2n + 1$.

Since $2n + 1 \neq 0$, this implies that $h(\phi)\text{ is non-trivial, and hence } h \text{ isn't nullhomotopic}$.

**Corollary:** There is no continuous antipode-preserving map $g : S^2 \rightarrow S^1$.

**Proof:** Suppose $g$ is continuous and antipode-preserving, & consider the equator $S^1 \subset S^2$.

Then $g|S^1 : S^1 \rightarrow S^1$ is an antipode-preserving map, hence not nullhomotopic by the previous then, however it extends to $B^2$ (northern hemisphere), contradiction.

$g|B^2 : B^2 \rightarrow S^1$

**Corollary (Borsuk-Ulam than for $S^2$):** If $f : S^2 \rightarrow R^2$ is continuous then $\exists x \in S^2$ s.t. $f(x) = f(-x)$.

**Proof:** Assume not, then $f(x) \neq f(-x)$ $\forall x$, so $g(x) = \frac{f(x) - f(-x)}{||f(x) - f(-x)||} : S^2 \rightarrow S^1$ is continuous and

$g(-x) = -g(x)$ $\forall x$, Contradiction.

**Corollary:** An open set in $R^2$ cannot be homeomorphic to an open set in $R^n$ for $n > 3$.

(For $R^2$ vs. $R^{n > 2}$ this is much easier: removing a point from $(a, b)$ disconnects it.)

**Proof:** Assume $U \subset R^n$ open and $f : U \rightarrow V \subset R^2$ homeo.

Then $\exists B_r(x) \subset U$ for some small $r > 0$, which is homeomorphic to $B^n \supset B^2 \supset S^2$.

So by restriction we get a continuous, injective map $f|_{S^2} : S^2 \rightarrow R^2$.

This contradicts Borsuk-Ulam.

**A fun application:**

Given a bounded polygonal (or "nice enough" - measurable suffice!) region $A \subset R^2$, $\exists$ straight line in $R^2$ that bisects it into equal area.

This is easy by intermediate value theorem:

If continuous, $\exists c$ s.t. $f(c) = \frac{1}{2} \text{ Area}(A)$.
Using Borsuk-Ulam, one shows:

**Theorem:**

Given two bounded (polyhedral) regions $A_1, A_2 \subset \mathbb{R}^2$, there exists a straight line in $\mathbb{R}^2$ that simultaneously bisects each of them into equal areas.

**Proof:**

Place $A_1, A_2$ in the plane $\mathbb{R}^2 \times \{1\} \subset \mathbb{R}^3$.

Given a unit vector $u \in S^2$, let $P \subset \mathbb{R}^3$ be the plane through origin with normal vector $u$. $P$ divides $\mathbb{R}^2$ into two half-spaces, and $\mathbb{R}^2 \times \{1\}$ into two half-planes (usually).

Let $f_i(u) = \text{area of the part of } A_i \text{ that lies on the side of } u$.

Note: $f_i(u) + f_i(-u) = \text{area}(A_i)$

Now, $F(u) = (f_1(u), f_2(u))$ is a continuous map $S^2 \to \mathbb{R}^2$

So, $\exists u \in S^2 \text{ s.t. } F(u) = F(-u) \implies f_i(u) = f_i(-u) = \frac{1}{2} \text{area}(A_i)$. □

This generalizes to: given $n$ bounded measurable sets in $\mathbb{R}^n$, there exists a hyperplane which bisects them all equally. In $\mathbb{R}^3$ this is called the “ham sandwich theorem”.

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\[\text{Diagram showing bisecting line and areas on both sides of the plane.}\]
Deformation retracts:

Recall for $AX$, $r : X \to A$ retraction = continuous map st. $r_A = \text{id}_A$.

Example:

- $S^1 \to S^2$ contact map
- $S^2 \to S^2^+$ (x,y,z) ↦ (x,y,|z|)
- $\mathbb{R}^2\backslash\{0\} \to S^1 \ x \mapsto x/|x|$
- Mobius band $\to S^1$

Important difference between the first two and the latter two: in the latter one, we can deform the identity map $\text{id}_X$ continuously in $X$ to get to the retraction.

Eg.

$X = \mathbb{I} \times I/(0,y) \sim (1,1-y)$  \quad $H : X \times I \to X$  \quad $H((x,y),t) = [x, t \cdot \frac{1}{2} + (1-t)y]$
(Whereas we can’t do this in first 2 example: eg we know $\text{id}_{S^1}$ is not null-homotopic).

Def. A subspace $AX$ is a deformation retract of $X$ if the identity map $\text{id}_X$ is homotopic to a map sending all $X$ into $A$, st. the points of $A$ are fixed during the homotopy.

ie: \exists continuous $H : X \times I \to X$ st.
- $H(x,0) = x \ \forall x \in X$
- $H(x,1) \in A \ \forall x \in X$
- $H(a,t) = a \ \forall a \in A \ \forall t \in I$

Such $H$ is called a deformation retraction of $X$ unto $A$.

The map $r : X \to A$ defined by $r(x) = H(x,1)$ is a retraction, and denoted by $i : A \to X$ the inclusion,

- $r \circ i = \text{id}_A$ (as usual for a retraction)
- $H$ is a homotopy between $\text{id}_X$ and $r \circ i : X \to X$

Example: $X = \mathbb{R}^n - \{0\}$, $H : X \times I \to X$ defined by $H(x,t) = t \frac{x}{|x|} + (1-t)x$

is a deformation retraction onto $S^{n-1}$.

If $a \in S^{n-1}$, $H(a,t) = t \frac{a}{t} + (1-t)a = a$ \quad (note: straight line segment from $x$ to $\frac{x}{|x|}$ doesn’t pass through 0)

\* We’ve seen before: if $A$ is a retract of $X$ then $i_\ast : \pi_1(A) \to \pi_1(X)$ is injective (because $r \circ i = \text{id}_A \Rightarrow r_\ast \circ i_\ast = \text{id}_{\pi_1(A)}$ so if $[f],[g] \in \pi_1(A)$ satisfy $i_\ast([f]) = i_\ast([g])$ then $[f] = i_\ast(i_\ast([f])) = r_\ast(i_\ast([f])) = r_\ast([g]) = [g]$).
Prop: \( h, k : (X, x_0) \to (Y, y_0) \) continuous maps. If \( h \) and \( k \) are homotopic with \( x_0 \mapsto y_0 \) throughout the homotopy, then \( h^* = k^* \).

PF: if \( f \) is a loop at \( x_0 \) in \( X \) then \( X \mapsto f \sim id \mapsto x \mapsto I \mapsto H \mapsto Y \)

\( H \circ (f \sim id) \) is a path homotopy between \( h \circ f \) and \( k \circ f \) in \( Y \).

Generalization: If base point doesn't stay fixed: let \( \alpha : \pi_1(Y, y_0) \to \pi_1(Y, y_1) \) induced by \( \alpha \) \( ([f] \mapsto [k \circ f \circ k^{-1}]) \), then \( k = \alpha \circ h_k \).

PF: now consider \( X \times I \mapsto X \times I \) left by concatenating \( \{ \text{path} (x_0, 0) \to (x_0, t) \}

\( \{ \text{loop} f \in X \times \{ 1 \} \}

\{ \text{path} (x_0, t) \to (x_0, 1) \} \)

then \( H \circ f \) is a path homotopy from

\( \alpha \circ (h \circ f) \circ \alpha \) to \( e \times (k \circ f) \circ e \).

Thus: if \( A \subset X \) is a deformation retract, then the inclusion \( i : (A, x_0) \to (X, x_0) \)

induces an isomorphism \( i_\# : \pi_1(A, x_0) \to \pi_1(X, x_0) \).

PF: let \( H : X \times I \to X \) deformation retraction, so \( H(x, 0) = x \), \( H(x, 1) \in A \), \( H(0, t) = \text{VaEA} \)

Denoting by \( r : X \to A \) the retraction \( R \) and \( i : A \to X \) the inclusion

\( r \circ i = \text{id}_A \), \( i \circ r = H_{|X \times \{ 1 \}} \) homotopic (via \( H \)) to \( \text{id}_X = H_{|X \times \{ 0 \}} \).

So \( r \circ i_\# = \text{id}_\pi(A) \), \( i_\# \circ r_\# = \text{id}_\pi(X) \); \( i_\# \) and \( r_\# \) are inverse isomorphisms.

Ex: 1) \( S' \) has the same \( \pi_1 \) as various spaces in which it is a deformation retract:

- \( S' \times I \) cylinder
- Möbius band
- \( B^2 \setminus \{ 0 \} \)
- \( S^1 \setminus B^2 \) solid torus

2) Figure 8 space \( S' \cup S' \) is a deformation retract of \( R^2 - \{ 2 \text{ points} \} \) (or \( B^2 - \{ 2 \text{ points} \} \))

& also of \( (S' \setminus \{ \text{point} \} \) (so these all have same \( \pi_1 \)).

So is the "theta graph" \( \bigcirc \) \( (S' \cup \{ (0) \} \setminus [-1, 1]) \subset R^2 \).

Yet neither of \( \bigcirc \) and \( \bigcirc \) is a deformation retract of the other!

So there's a more general situation under which \( \pi_1 \) (and "all of homotopy theory") remains unchanged.
Homotopy equivalence:

**Def.** Let \( f: X \to Y \) and \( g: Y \to X \) be continuous maps. If \( g \circ f: X \to X \) is homotopic to \( \text{id}_X \) and \( f \circ g: Y \to Y \) is homotopic to \( \text{id}_Y \), then \( f \) and \( g \) are homotopy equivalences, and we say \( X \) and \( Y \) have the same homotopy type. \( f \) and \( g \) are homotopy inverses of each other.

**Ex.** If \( A \) is a deformation retract of \( X \), then \( X \overset{\iota}{\to} A \) is homotopic to \( \text{id}_X \), and \( A \overset{\iota'}{\to} X \) is identity. So \( X \) and \( A \) have the same homotopy type.

**Ex.** \( A = \infty \quad c \mapsto \begin{array}{c} \infty \\ X = \mathbb{R}^2 - 2 \text{ pts} \end{array} \rightarrow \begin{array}{c} \infty \\ \iota' \end{array} = A' \quad \text{and vice versa.} \)

Then \( A \overset{i}{\to} X \overset{f}{\to} A' \overset{i'}{\to} X \overset{g}{\to} A \) is homotopic to \( A \overset{i}{\to} X \overset{g}{\to} A = \text{id}_A \).

This is true more generally!

**Prop.** If \( f: X \to Y \) and \( g: Y \to Z \) are homotopy equivalences, then \( g \circ f: X \to Z \) is a homotopy equivalence. Thus, having the same homotopy type is an equivalence relation.

(Some proof as in the example—the composition of homotopy inverses to \( f \) and \( g \) give a homotopy inverse to \( g \circ f \)).

**Ex.** Recall \( X \) is contractible if \( \text{id}_X \) is homotopic to a constant map \( X \to \{p\} \) for some \( p \in X \) (e.g. \( I, B^n, \mathbb{R}^n \), etc.). Then \( \{p\} \subset X \to \{p\} \) is identity and \( X \to \{p\} \overset{c}{\to} X \) is homotopic to \( \text{id}_X \), so \( \{p\} \subset X \) is a homotopy equivalence.

(Note: the given homotopy \( H \) between \( \text{id}_X \) & cont. map does not necessarily give a deformation retraction, since \( p \) might not be fixed by the homotopy).

**Thm.** If \( f: (X, x_0) \to (Y, y_0) \) is a homotopy equivalence, then \( f_\#: \pi_1(X, x_0) \approx \pi_1(Y, y_0) \) isomorphism.

**Pf.** Let \( g: Y \to X \) be a homotopy inverse for \( f \), with \( g(y_0) = x_0 \). Then we have maps:

\[
\pi_1(X, x_0) \xrightarrow{f_\#} \pi_1(Y, y_0) \xrightarrow{g_\#} \pi_1(X, x_0) \xrightarrow{f_\#} \pi_1(Y, y_1)
\]
$g \circ f$ is homotopic to $id_x$, so $\exists$ path $\alpha$ from $x_0 = id(x_0)$ to $x_1 = g \circ f(x_0)$, and $g \circ f = \hat{\alpha} \circ id_n(x) = \hat{\alpha}$, an isomorphism. (hence surjective)

using generalization of statement about homotopic maps inducing same hom on $\pi_1$.

Similarly, $f'_k \circ g_k : \pi_1(\Sigma X, x_0) \to \pi_1(\Sigma X, y_1)$ is an isomorphism. (hence injective)

These imply that $g_k$ is injective & surjective, hence an isomorphism, and hence $f_k = (g_k)^{-1} \circ \hat{\alpha}$ is also an isomorphism. \qed
Last time: homotopy equivalence i.e. a pair of maps \( X \xrightarrow{f} Y \) s.t. \( g \circ f \sim 1_Y \) under \( g(y_0) = x_0 \) & homotopic preserve the base points, the proof is:
\[
\begin{align*}
g \circ f & \sim 1_Y \\
f \circ g & \sim 1_X
\end{align*}
\]

Removing the assumption on base points required some further work involving isomorphisms induced by moving base point along a path.

This gave access to many calculations of \( \pi_1 \).

**Ex**: \( \mathbb{C} \times \mathbb{R} \) deformation retract \( (i : A \to X, r : X \to A, r \circ i = \text{id}_A, i \circ r \sim \text{id}_X \text{ fixing } A) \).

All contain \( S^1 \) as a deformation retract, \( \pi_1 \cong \mathbb{Z} \).

**Ex**: \( X \) is contractible if \( \text{id}_X \) is homotopic to a constant map. Then \( \{ p \} \rightarrow X \) is a homotopy equivalence, and \( \pi_1(X) = 1 \).

**Goal**: More tools for calculations of fundamental groups. Start with: \( (559-60) \)

**Q**: Assume \( X = U \cup V \), with \( U \) and \( V \) open subsets, and we know \( \pi_1(U) \) and \( \pi_1(V) \). Can we find \( \pi_1(X) \)?

**Eq**: \( S^2 = U \cup V \). \( \pi_1(U) \) & \( \pi_1(V) \) trivial

Figure 8 = \( U \cup V \), each of \( U \) & \( V \) has homotopy type of \( S^1 \).

The Seifert-Van Kampen, which we'll see later, gives a general way to calculate \( \pi_1(X) \) in this situation. For now we'll just prove a weaker (and easier) version.

**Thm**: Suppose \( X = U \cup V \), \( U \) and \( V \) open, \( U \cap V \) path-connected, \( x_0 \in U \cap V \).

Let \( i : U \to X \) and \( j : V \to X \) be the inclusion maps. Then the images of \( i_* : \pi_1(U, x_0) \to \pi_1(X, x_0) \) and \( j_* : \pi_1(V, x_0) \to \pi_1(X, x_0) \) generate \( \pi_1(X, x_0) \).

The statement means: every element of \( \pi_1(X, x_0) \) can be expressed as a product of elements in \( \text{Im}(i_*) \) and \( \text{Im}(j_*) \) — i.e. every loop in \( (X, x_0) \) is path-homotopic to a composition of loops at \( x_0 \) which are entirely contained in either \( U \) or \( V \).
Let \( f: I \to X \) be a loop based at \( x_0 \).

\[ [0,1] = f^{-1}(U) \cup f^{-1}(V) \text{ open cover, } [0,1] \text{ compact} \]

\Rightarrow using the Lebesgue number lemma, we can subdivide \([0,1]\) into \( 0 = a_0 < a_1 < \ldots < a_n = 1 \) st. \( f([a_{i-1}, a_i]) \) is contained in either \( U \) or \( V \). Eliminating unnecessary \( a_i \) from the list, can assume \( U \) and \( V \) alternate along the way, and in particular \( f(a_i) \in U \cup V \).

Let \( f_i = f|_{[a_{i-1}, a_i]} \) so that \( [f] = [f_1] \ast \ldots \ast [f_n] \).

For each \( i \), choose a path \( \alpha_i \) in \( U \cup V \) from \( x_0 \) to \( f(a_i) \).
(take \( \alpha_0 = \alpha_n = \text{constant path at } x_0 \)).

Then \( [f] = [\alpha_0 \ast f_1 \ast \alpha_1^{-1}] \ast [\alpha_1 \ast f_2 \ast \alpha_2^{-1}] \ast \ldots \ast [\alpha_{n-1} \ast f_n \ast \alpha_n^{-1}] \)

loops at \( x_0 \), entirely contained in \( U \alpha \) in \( V \)

**Corollary:**
\[ X = U \cup V \text{ with } U \cup V \text{ open and simply-connected } \Rightarrow X \text{ is simply-connected.} \]

**Example:**
Let \( X = S^n \), \( n \geq 2 \), and \( U = S^n - (0,0,\ldots,0) \) \( N \) North pole
\( V = S^n - (0,\ldots,0,1) \) \( S \) South pole.

Then \( U \) and \( V \) are homeomorphic to \( \mathbb{R}^n \) via stereographic projection \( f: U \to \mathbb{R}^n \)

mapping each point \( x \in U \) to the point where the line in \( \mathbb{R}^{n+1} \) through \( N \) and \( x \) intersects the equatorial plane \( \mathbb{R}^n \times \{0\} \).

I.e.: \( f(x) = \frac{1}{1 - x_{n+1}} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \)

\( 1 \) change to \( + \) for \( V \Rightarrow \mathbb{R}^n \).

Hence, \( U \) and \( V \) are homeomorphic to \( \mathbb{R}^n \), are simply connected
\( U \cup V \text{ is path-connected } (n \geq 2!) \)

**Corollary:**
\( S^n \) is simply connected for \( n \geq 2 \).

**Example:**
recall from HW8, the quotient of \( S^n \) by \( x \sim -x \), \( p: S^n \to S^n/\sim \approx \mathbb{R}^n \) is a degree 2 covering map.
Also recall: lifting commonadence \( \pi_1(\mathbb{R}P^n, b_0) \rightarrow p^*\{b_0\} = \{2 \text{ points}\} \)

subjective because \( S^n \) connected; injective because \( S^n \) is simply connected.

(If a loop \( f \) in \( \mathbb{R}P^n \) lifts to a loop \( \tilde{f} \) in \( S^n \), then \( \tilde{f} \) is homotopic to constant loop in \( S^n \).

& projecting by \( p \), \( p \circ \tilde{f} = f \) is homotopic to a constant loop in \( \mathbb{R}P^n \)).

So \( \pi_1(\mathbb{R}P^n) \) is a group with 2 elements, hence isomorphic to \( \mathbb{Z}/2\mathbb{Z} \).

Ex.: \( X \) = figure 8 space, \( \begin{array}{c}
\begin{array}{c}
\text{can cover by open sets } U, V \text{ which have deformation retractions to } S^1, \ U \cap V = \emptyset \text{ bounded}
\end{array}
\end{array} \)

By theorem, \( \pi_1(X) \) is generated by the image of two maps \( x, y \),

i.e. can express every loop in terms of powers of \( [a] \) and \( [b] \) (\( a, b \) loops around each circle).

which generate \( \pi_1(U), \pi_1(V) \), i.e. every element is of the form \( \prod_{i \in I} [h_i]^{n_i} \),

but don't know relations between \([a] \) and \([b] \).

Can show that \([a] \) and \([b] \) don't commute - \([a]*[b] \neq [b]*[a] \).

One way to do this is by looking at covering map

\[ \begin{array}{c}
\begin{array}{c}
\text{The lift of } a \times b \text{ starting at } e_0 \text{ ends at } (1, 0) \text{ hence } [a]*[b] \neq [b]*[a]
\end{array}
\end{array} \]

so \( \pi_1(X, x_0) \) is not abelian. In fact, we'll show later that it is the free group generated by \([a] \) and \([b] \), i.e. \( \text{elts are arbitrary words in } [a]^{\pm 1} \text{ and } [b]^{\pm 1} \)

with no relations whatsoever (except \([a]^{-1}*[a] = 1 \) etc.).

Now about free groups later.
Q: Let $p : (E, e_0) \to (B, b_0)$ covering map. How are $\pi_1(E)$ and $\pi_1(B)$ related?

(Always assume $E$ and $B$ are path-connected).

Then: $p_* : \pi_1(E, e_0) \to \pi_1(B, b_0)$ is an injective homomorphism.

Pf: if $\tilde{t}$ is a loop at $e_0$ and $p_0([\tilde{t}]) = \text{id}$, then $\exists$ path homotopy $H : I \times I \to B$ from $p_0 \tilde{t}$ to the constant loop at $b_0$. Its lift $\tilde{H} : I \times I \to E$ starting at $e_0$ is then a path homotopy from $\tilde{t}$ to the constant loop, so $[\tilde{H}] = \text{id}$.

Hence, each covering $p : E \to B$ gives a subgroup $H \subset \pi_1(B, b_0)$ that is isomorphic to $\pi_1(E, e_0)$ via $p_*$.

It turns out that:

1. The subgroup $H \subset \pi_1(B, b_0)$ determines the covering $p$ up to "equivalence".
2. Assuming $B$ is path-connected and "sufficiently nice" ("semi-locally simply connected"), for each subgroup $H$ of $\pi_1(B, b_0)$ $\exists$ covering $p : E \to B$ s.t. $p_0(\pi_1(E)) = H$.

We'll focus on (1) now; this is Munkres §53. ("Equivalence of covering spaces")

We'll prove (2) in this class - see Munkres §82. ("Existence of covering spaces")

---

**Equivalence of covering spaces**:

**Def**: $p : E \to B$, $p' : E' \to B$ coverings. $p$ and $p'$ are equivalent if $\exists$ homomorphism $h : E \to E'$ s.t. $p = p' \circ h$. Say $h$ is an equivalence of coverings $p \cong p'$.

**NB**: $\forall b \in B$, $h$ gives a bijection $p'(b) \cong p''(b)$, and in fact given $U = \cap V \supset b$ evenly covered, must map $p'(U) \cong U \times A$ to $(p'')^{-1}(U) \cong U \times A'$ via a bijection $A \to A'$ on set of sheets.

**Ex**: $p(x) = (\cos x, \sin x) : \mathbb{R} \to S^1$ $p'(x) = (\cos(2\pi x), \sin(2\pi x))$ equivalent via $h(x) = 2\pi x$.

- Goals: if two coverings have same computing subgroup of $\pi_1(B)$ then they are equivalent.
- For this we need a general lifting lemma.

**Def**: A space $X$ is locally path-connected if $\forall x \in X, \forall U \ni x, \exists V \subset U$ path-connected neighborhood of $x$. 

Counterexample: \((\{x_n\}_{n=1}^{\infty} \cup \{0\}) \times \mathbb{R} \cup \mathbb{R} \times 0 \text{ in } \mathbb{R}^2\)

is path-connected but not locally path-connected.

From now on, assume \(p: E \to B\) covering, \(E\) and \(B\) path-connected and locally path-connected.

Lifting lemma for loops:

Then \(| A \text{ loop in } (B, b_0) \text{ lifts to a loop in } (E, e_0) \iff [F] \in \pi_1(E, e_0) \cap \pi_1(B, b_0) | \]

Proof:
- If the lift \(\tilde{F}\) of \(f\) at \(e_0\) is a loop in \(E\), then \([F] = [\tilde{F}] = [p \circ \tilde{F}] \in \pi_1(E, e_0)\).
- If \([F] = p_\#([\tilde{F}])\) for some loop \(\tilde{F}\) in \((E, e_0)\), then \(p \circ \tilde{F}\) is path-homotopic to \(f\).

Lifting this path-homotopy to \(E\), we get a path-homotopy in \(E\) between \(\tilde{F}\) and the lift \(\tilde{F}\) of \(f\). Since \(\tilde{F}\) is a loop, so is \(F\).

General lifting lemma:

Then \(| \text{Let } p: E \to B \text{ covering map, } p(e_0) = b_0. \text{ Let } Y \text{ be path-connected & loc.-path-connected, } | \]

and \(f: Y \to B\) continuous map s.t. \(f(y_0) = b_0\). Then \(f\) can be lifted to \(\tilde{f}: Y \to E\) with \(\tilde{f}(y_0) = e_0 \iff f_\#(\pi_1(Y, y_0)) \subseteq p_\#(\pi_1(E, e_0)). \text{ If it exists, the lift is unique |} \]

Proof:
- If \(f\) can be lifted to \(\tilde{f}\), then \(\tilde{f} = p \circ \tilde{f}\), so \(f_\#(\pi_1(Y, y_0)) \subseteq p_\#(\pi_1(E, e_0))\).

Conversely, assume the condition holds, and let \(y_0 \in Y\). Choose a path \(\alpha\) from \(y_0\) to \(y_1\) in \(Y\). Lift \(\alpha\) to \(E\) to a path in \(E\) starting at \(e_0\).

Define \(\tilde{f}(y) = \text{ the end point of this path }\).

(This is the only possibility for \(\tilde{f}(y)\) if a continuous lift \(\tilde{f}\) exists, since the unique lift of \(\alpha\) will then be \(p \circ \tilde{f}\).)

Need to check \(\tilde{f}\) is well-defined and continuous!

Well-defined? Let \(\beta\) be a different path in \(Y\) from \(y_0\) to \(y_1\).

Then \(\alpha \ast \overline{\beta}\) is a loop in \((Y, y_0)\)

\(f_\#(\alpha \ast \overline{\beta}) \text{ loop in } (B, b_0)\), representing \(f_\#(\alpha \ast \overline{\beta}) \in [F] \subseteq p_\#(\pi_1(E, e_0))\)

so it lifts to a loop in \(E\) (by previous theorem).

So: \(f_\# \alpha \text{ lifts to a path from } e_0 \text{ to } \tilde{f}(y_1) \text{ as defined above, and } f_\# \overline{\beta} \text{ lifts to a path from } \tilde{f}(y_1) \text{ back to } e_0, \text{ hence } f_\#(\alpha \ast \overline{\beta}) \text{ lifts to a path from } e_0 \text{ to } \tilde{f}(y_1)\). Thus \(\tilde{f}(y)\) is independent of the choice of \(\alpha \text{ and } \beta\).
Continuity of $\tilde{f}$: enough to check on a neighborhood of $y_1$. 
Let $V \subset B$ be an evenly covered nbhd of $f(y_1)$, and using local path-connectedness of $Y$, can find $U = f^{-1}(V)$ path-connected neighborhood of $y_1$ in $Y$.
Let $W = f^{-1}(V) \subset E$ be the slice containing $\tilde{f}(y_1)$; $p|_W = \pi; W \cong V$ homeo.

For $y \in U$, $\exists$ path $\gamma$ in $U$ from $y_1$ to $y$, and $\pi_1^{\gamma}f|_\gamma$ is a lift of $f|_\gamma$ to $W \subset E$ starting at $\tilde{f}(y_1)$. And so the lift of $f_0(\alpha \times \gamma)$ to $E$ starting at $e_0$ is the composition of $f_0 \circ (\gamma)$ (from $e_0$ to $\tilde{f}(y_1)$) and $\pi_1^{\gamma}f|_\gamma$ from $\tilde{f}(y_1) = \pi_1^{\gamma}(f(y_1))$ to $\pi_1^{\gamma}(f(y))$. Hence $\tilde{f}(y) = \pi_1^{\gamma}(f(y))$.
So $\tilde{f}|_U = \pi_1^{\gamma}f|_U$ is continuous, and hence $\tilde{f}$ is continuous.

Now we can tell when two coverings are equivalent, as long as all maps preserve base points!

**Theorem:** Let $p : E \to B$, $p' : E' \to B$ covering maps with $p(e_0) = p'(e'_0) = b_0$.
There is an equivalence $h : E \cong E'$ s.t. $h(e_0) = e'_0$
if and only if the subgps $H = p_*(\pi_1(E, e_0))$ and $H' = p'_*(\pi_1(E', e'_0))$
are equal (the same subgroup of $\pi_1(B, b_0)$).
Moreover, if $h$ exists it is unique.

**Proof:** if $h : E \to E'$ is an equivalence with $h(e_0) = e'_0$, then $h_*(\pi_1(E, e_0)) = \pi_1(E', e'_0)$.

The conclusion then follows from $p' \circ h_* = p_*$.

$\iff$ assume $H = H'$. Then by the lifting lemma, $\exists$ unique base point preserving lifts $h : E \to B$ and $h' : E' \to B$. So $p' \circ h = p$ and $p \circ h' = p'$.

Now, $p \circ h' \circ h = p \circ h = p$, so $h' \circ h : E \to E$ is a lifting $E \to B$.

But so is $i_\epsilon$. By uniqueness of lifting, we get $h' \circ h = i_\epsilon$.
Similarly $h \circ h' = i_{E'}$. So $h$ is a homeomorphism s.t. $p \circ h = p$, hence an equivalence of coverings.
$\text{Ex: }\ p_k: S^1 \rightarrow S^1 \quad (p_k)_x: \pi_1(S^1, b_0) \rightarrow \pi_1(S^1, b_0) = \mathbb{Z} \quad \text{compacts to null by } k,$

so the compactly submers is

$H_k = k\mathbb{Z} \subset \mathbb{Z}.$

Then are all the subgroups of $\mathbb{Z},$

so every connected covering of $S^1$ is

equivalent to exactly one of these!

---

What if we consider equivalence $h: E \rightarrow E'$ that don't map $e_0$ to $e'_0$?

A: next time.
Recall: $p : E \to B$, $p' : E' \to B$ covering maps. $p$ and $p'$ are equivalent if there exists a homeomorphism $h : E \to E'$ such that $h(p(e)) = p'(e_0) = b_0$. Say $h$ is an equivalence of covering maps $p \overset{h}{\to} p'$.

Then: let $p : E \to B$, $p' : E' \to B$ covering maps with $p(e_0) = p'(e_0) = b_0$. There is an equivalence $h : E \to E'$ such that $h(e_0) = e_0'$ if and only if the subgroups $H = p_*(\pi_1(E, e_0))$ and $H' = p'_*(\pi_1(E', e_0'))$ are equal (the same subgroup of $\pi_1(B, b_0)$).

Moreover, if $h$ exists, it is unique.

What if we consider equivalence $h : E \to E'$ that don't map $e_0$ to $e_0'$?

Then the corresponding subgroups of $\pi_1(B, b_0)$ are conjugate.

Indeed, if we change the base point in a (path-connected) covering space $p : E \to B$...

If $e_0, e_1 \in p'(b_0)$, and $\alpha$ is a path from $e_0$ to $e_1$, recall $\pi_1(E, e_0) \to \pi_1(E, e_1)$ 
$[h] \mapsto [\alpha^* \times h \times \alpha]$.

Then $\alpha = p \circ \alpha'$ is a loop in $(B, b_0)$, so whenever $[p \circ h] = p_*([h]) \in H_0 = p_*([\pi_1(E, e_0)])$ implies $[\alpha'^{-1} \times [p \circ h] \times [\alpha] \in H_1 = p'_*(\pi_1(E, e_1))$.

So, $\lbrack \alpha'^{-1} \rbrack H_0 \lbrack \alpha \rbrack \subset H_1$, and similarly in the reverse direction $\lbrack \alpha \rbrack H_1 \lbrack \alpha'^{-1} \rbrack \subset H_0$, so $\lbrack \alpha \rbrack H_0 H_1 \lbrack \alpha'^{-1} \rbrack = H_1$.

Conversely, if $H_0, H_1$ are conjugate subgroups of $\pi_1(B, b_0)$, i.e., $\exists \alpha$ such that $H_1 = [\alpha]^{-1} H_0 [\alpha]$ and $H_0 = p_*(\pi_1(E, e_0))$, then let $\tilde{\alpha}$ be lift of $\alpha$ to a path in $E$ starting at $e_0$, and let $e_1 = \tilde{\alpha}(1)$, then $H_1 = p_*(\pi_1(E, e_1))$.

Theorem: $p : E \to B$, $p' : E' \to B$ covering maps, $p(e_0) = p'(e_0) = b_0$. Then $p$ and $p'$ are equivalent if the subgroups $H = p_*(\pi_1(E, e_0))$, $H' = p'_*(\pi_1(E', e_0'))$ of $\pi_1(B, b_0)$ are conjugate.

Proof: if $h : E \to E'$ is an equivalence, with $e_0' = h(e_0)$, let $G = p'_*(\pi_1(E', e_0'))$. Then $G = H$ by previous theorem, and $G$ is conjugate to $H'$ by above discussion.

If $H$, $H'$ are conjugate, then $\exists e_1' \in E'$ such that $H = p'_*(\pi_1(E', e_1'))$ by above discussion.

Then by the previous theorem, $\exists$ equivalence $h : E \to E'$ with $h(e_0) = e_1'$. \qed
Universal covering space:

Def: If \( p_0: E_0 \to B \) is a covering and \( E_0 \) is simply connected, say \( E_0 \) is a universal covering of \( B \).

Note: this corresponds to the trivial subgroup \( p_0^*(\pi_1(B)) = \{1\} \subseteq \pi_1(B) \), unique up to equivalence by the above.

Ex: \( p: R \to S^1 \)

- \( p \) is a universal covering.
- \( p: R \to S^1 \times S^1 \) is torus

Then: \( p_0: E_0 \to B \) universal covering, \( p': E' \to B \) any path-connected covering, then

- \( \exists \) covering map \( q_0: E_0 \to E' \) s.t. \( p_0 q_0 = p' \) and \( q_0 \) is universal covering of \( E' \).

- \( q_0 \) is contractible by lifting:

\[
\begin{align*}
E_0 & \xrightarrow{q_0} E' \\
E_0 & \xrightarrow{p_0} B
\end{align*}
\]

\( \exists \) since \( p_0^*(\pi_1(E_0)) = \{1\} \subseteq p_0^*(\pi_1(E')) \).

So, in fact, if \( B \) has a universal covering, all other coverings can then be obtained as quotients!

- Some spaces have no universal covering:

Ex: "Hawaiian earrings" = \( \bigcup \overset{\text{n=1}}{\overset{n}{C_n}} \) circle of radius \( \frac{1}{n} \) centered at \( (\frac{1}{n}, 0) \) inside \( R^2 \).

- Any covering space must evenly cover a neighborhood of the origin, which prevents it from being simply connected. (For a sufficiently large, loop around \( C_n \) lifts to a loop).

- If one avoids such pathological examples - assuming \( B \) is (semi-)locally simply connected, can build universal covering as space of pairs \((b, \hat{\gamma})\) where \( \{b \in B \mid \hat{\gamma} = \text{homotopy class of path } b \to b\} \)

This has a preferred topology for which any simply connected \( U \supseteq B \) is evenly covered:

- If \( b' \in U \), adding a path \( b \to b' \) inside \( U \) or its inverse gives a preferred bijection \( \{\text{homotopy classes of paths } b_0 \to b\} \leftrightarrow \{\text{homotopy classes of paths } b_0 \to b'\} \) independent of choice of path \( b_0 \to b' \) inside \( U \) since \( U \) simply connected.

Free products & free groups (pre: work for Van Kampen).

Assume \( G \) is a group, \( G_1 \ldots G_k \) subgroups of \( G \) which generate \( G \), i.e. any \( x \in G \) can be written as \( x = x_1 \ldots x_n \) where each \( x_i \) is in some \( G_j \). Also assume \( G_j \cap G_k = \{1\} \forall j \neq k \).

\((x_1, \ldots, x_m)\) is called a word of length \( m \) that represents \( x \).

Say \((x_1, \ldots, x_m)\) is a reduced word if no \( G_j \) contains two consecutive elements \( x_i, x_{i+1} \).

(in particular if \( m = 2 \), no \( x_i \) can be \( \equiv 1 \)). (Else reducible to a shorter word \((x_1, x_{i+1}, \ldots, x_m)\)).
Def: $G$ is the free product of the subgroups $G_1 \ldots G_n$, denoted $G = G_1 \ast \ldots \ast G_n$, if $G_i$ generate $G$, $G_1 \cap G_2 = \{1\}$, and $\forall x \in G$ there is only one reduced word that represents $x$.

Ex: $\mathbb{Z} \ast \mathbb{Z}$ is not the free product of its two factors: denoting by $a$ and $b$ the two generators $(a = (1,0), b = (0,1))$, $ab = ba$ is repeated by several reduced words: $(a,b)$, $(b,a)$, but also $(a^2, b, a^{-1})$, etc.

1. Alternative characterization: $G$ is the free product of the subgroups $G_j$'s iff, for any group $H$ and any homomorphisms $h_j: G_j \to H$, $\exists$ unique homomorphism $h: G \to H$ s.t. $G_j \to G \xrightarrow{h} H$ commutes $h_j$.

(The point is: uniqueness of expression allows us to define $h(x_1 \ldots x_n) = h_j(x_1) \ldots h_j(x_n)$ for each $x_i \in G_j$)

2. External free product of groups $G_j :=$ group $G$ + injective hom's $G_j \rightarrow G$ st. $G$ is the free product of the subgroups $\pi_j(G_j)$

Fact: This always exists! 2. unique up to iso.

1. Can be constructed as set of reduced words in $G_j$'s (with product = concatenate + reduce) & satisfies universal property (1)

2. In particular the free group on the elements $\{a_j\}$ is defined to be the external free product of groups $G_j = \{a_j^n | n \in \mathbb{Z}\}$ ($a^j \cdot a^k = a^{j+k}$, so $G_j \cong \mathbb{Z}$)

(so: set of (reduced) words consisting of powers of $a_j$'s)

Why do we care?

A: We've seen before: $X = U \cup V$, $U, V$ open, $U \cap V$ path-connected $\Rightarrow x_0 \Rightarrow \pi_1(X, x_0)$ is generated by the subgroups $\pi_1(U, x_0)$ and $\pi_1(V, x_0)$.

1. If additionally $U \cap V$ is simply-connected, then $\pi_1(X, x_0)$ is the free product of the two subgroups $j_{\pi_1(U, x_0)}$ and $j_{\pi_1(V, x_0)}$. (= external free product of $\pi_1(U) \ast \pi_1(V)$, as is $\cup$ in fact injective).

(This is a special case of Van Kampen's theorem)

Example: $\pi_1(\bigcirc \bigcirc) = \mathbb{Z} \ast \mathbb{Z}$ free group on 2 generators $a$ and $b$.

(more generally, for wedge of $n$ circles, free group on $n$ letters)
Serre-Van Kampen: Let \( X = U \cup V \), \( U \) and \( V \) open in \( X \), \( U \cap V \) path-connected \( \exists x_0 \).

The inclusions \( U \cap V \to X \) induce homomorphisms on \( \pi_1 \).

\[
\begin{array}{ccc}
\pi_1(U, x_0) & \xrightarrow{h} & \pi_1(X, x_0) \\
\pi_1(V, x_0) & \xrightarrow{\text{concatenation}} & \pi_1(X, x_0)
\end{array}
\]

By universal property of free product, \( \exists \) unique homomorphism \( h \)

\[
h \cdot \pi_1(U, x_0) \cdot \pi_1(V, x_0) \cong \pi_1(X, x_0)
\]

(Define \( h \) on words in elements of \( \pi_1(U, x_0) \) and \( \pi_1(V, x_0) \) using \( i_1 \) and \( i_2 \)

\((\text{define } h \text{ on words in elements of } \pi_1(U, x_0) \text{ and } \pi_1(V, x_0) \text{ using } i_1 \text{ and } i_2 \text{ on each component of the word!})\)

Then (Serre-Van Kampen):

The homomorphism \( h \) defined above is surjective, and its kernel \( N \) is the

smallest normal subgroup of \( \pi_1(U, x_0) \times \pi_1(V, x_0) \) which contains all elements of the

form \( i_1^{-1}(g) \cdot i_2(g) \) \( \forall g \in \pi_1(U \cap V, x_0) \). I.e., \( \pi_1(X, x_0) \cong \pi_1(U, x_0) \times \pi_1(V, x_0) / N \).

Corollary 1: if \( U \cap V \) is simply connected then \( \pi_1(X, x_0) \cong \pi_1(U, x_0) \times \pi_1(V, x_0) \).

Corollary 2: if \( V \) is simply connected then \( \pi_1(X, x_0) \cong \pi_1(U, x_0) / N \), where

\( N \) is the smallest normal subgroup containing the image of \( i_1 : \pi_1(U \cap V, x_0) \to \pi_1(U, x_0) \).

Ex. 1: Figure 8: \( \bigcirc \bigcirc \xrightarrow{x_0} \Rightarrow U, V \) deformation retract onto circles

\( U \cap V \) contractible

Hence \( \pi_1(X, x_0) \cong \pi_1(U, x_0) \times \pi_1(V, x_0) \cong \mathbb{Z} \times \mathbb{Z} \) free group generated by loops around the two circles.

Ex. 2: by inductive wedge of \( n \) circles: \( X = \bigcup_{i=1}^{n} S_i \), \( S_i \) homeo to \( S^1 \) \( \forall i \), \( S_i \cap S_j = \{x_0\} \).

\( \Rightarrow \pi_1(X, x_0) = \text{free group on } n \text{ generators } a_i = \text{loops generating } \pi_1(S_i, x_0). \)
Fact: any (connected) finite graph (= union of intervals glued at end points) has the homotopy type of a wedge of finitely many circles.

Idea: contracting by an edge that connects different vertices is a homotopy equivalence as in

\[ \text{set all these } \sim \]

Proceed inductively until only one vertex remains.

get a homotopy eq to \( \bigvee S^1 \)'s

(Can also consider infinite wedge of circles - note this is not homeo to Hawaiian earrings)

\[ \bigcup_{n=1}^{\infty} C_n \neq \bigcup_{n=1}^{\infty} C_n / \sim \]

\( C_r \) radical circle \( @ (r,0) \). 

---

Fundamental groups of surfaces can also be calculated using Van Kampen!

eg. can now calculate \( \pi_1 \) of torus in a different way:

\[ T = \mathbb{I} \times \mathbb{I} / (x,0) \sim (x,1) \quad \forall x \\ (0,y) \sim (1,y) \quad \forall y. \]

Let \( U = T \setminus \{p\} \)

\[ V = \text{open ball of radius } < \frac{1}{2} \text{ around } p. \]

\( U \) deformation retracts onto wedge of two circles

\( V \) is simply connected.

\( U \cup V \sim D^2 - pt \) has homotopy type of \( S^1 \).

Using Corollary 2 above: \( \pi_1(T) \cong \pi_1(U)/N \) where \( N \) is normal generated by the image of the loop \( f \) which generates \( \pi_1(U \cup V) \) (and its conjugates)

\( \pi_1(U) \) is a free group on gens. \( a,b \) and then the image of \( [f] \) under the inclusion \( U \cup V \hookrightarrow U \) is \( aba^{-1}b^{-1} \)

\[ [\text{the "obvious" picture needs to be corrected slightly}: \] 

[basepoint should be fixed \( \in U \cup V \)]

So we set \( aba^{-1}b^{-1} = 1 \) i.e. \( ab = (aba^{-1}b^{-1})ba = ba \), get abelian group \( \mathbb{Z}^2 \)

\[ \pi_1(T) \cong \langle a,b | ab = ba \rangle \cong \mathbb{Z} \times \mathbb{Z}. \]

- Similarly for \( \pi_1(\mathbb{R}P^2) \), using \( \mathbb{R}P^2 \cong S^2 / \sim \) \( \xrightarrow{\cong} B^2 / \sim \)

\[ \xrightarrow{\sim} \mathbb{R} \times \mathbb{R} \setminus \{0\} = \mathbb{R}^2 \]
Now write $\mathbb{RP}^2 = U \cup V$, $U = \mathbb{RP}^2 - \{p\}$

$$V = d \circ c \text{ centered at } p$$

$U$ deformation retracts onto the boundary $S^1/\sim \rightarrow S^1$

so $\pi_1(U) \cong \mathbb{Z}$ w/ generator $c$

$V$ is simply connected. $U \cap V \cong \mathbb{D}^2$ - pt has homotopy type of $S^1$

$\pi_1(\mathbb{RP}^2) \cong \pi_1(U)/N$, $N$ normal subgroup generated by image of generator $[f] \in \pi_1(U \cap V)$ under inclusion, which is $c^2$.

So $\pi_1(\mathbb{RP}^2) \cong \langle c \mid c^2 = 1 \rangle \cong \mathbb{Z}/2\mathbb{Z}$.

**Klein bottle:** recall $K = \mathbb{R}^2 \cup \mathbb{R}^2/\sim$

$$(x,0) \sim (x,1), (0,y) \sim (1,1-y)$$

Again write $K = U \cup V$, $U = K - \{p\}$

$\pi_1(U) \cong \mathbb{Z}$ under inclusion, which is $a,b$.

$\pi_1(U \cap V)$ free group on generators $a,b$.

$\pi_1(K) \cong \pi_1(U)/N$ maps under inclusion to $aba^2b$.

So $\pi_1(K) \cong \langle a,b \mid aba^2b = 1 \rangle$ not abelian: $ab = b^2a$, not $ba$!

i.e. $aba^2b^{-1} = b^{-2}$ : b conjugate to its inverse!

But this contains an index 2 subgroup H gen'd by $a^2$ and $b$, which commute!

$$\langle aba^2b^{-1} \rangle \text{ taking inverses, } ab^{-1}a^{-1} = b, \text{ so } a^{-1}ba^{-2} = a (aba^{-1})a^{-1} = ab^{-1}a^{-1} = b$$

So $a^2b = ba^2 \checkmark$. ($\Rightarrow$ subgroup $H \cong \mathbb{Z}^2$).

Can show, by rearranging letters via $ab = b^2a$, this contains all words with even # of a's so it is an index 2 subgroup.

This subgroup corresponds to a deck 2 covering map by the torus, $T \rightarrow K$

I.e. map $(x,y) \in \mathbb{I} \times \mathbb{I}/\sim_T$ to

$$\{(2x,y) \text{ if } x \leq \frac{1}{2}, (2x-1,1-y) \text{ if } x > \frac{1}{2} \} \text{ in } \mathbb{I} \times \mathbb{I}/\sim_K.$$

Cool fact that this relates to: if you coat a klein bottle in paint all over, the paint forms a torus.
1. **Topological spaces**: \((X, T)\), \(T = \{ U \subseteq X | U \text{ open} \}

- **Axioms**: \(\emptyset, X\), arbitrary unions, finite intersections.
- \(F \subseteq X \text{ closed} \iff F \subseteq X \text{ open}.

- **Basis for a topology**: \(B\)
  - Open sets = union of elements of \(B\).
  - \(U_B = X; x \in B \cap B' \Rightarrow x \in B' \cap B\).
  - \(U \text{ open} \iff \forall x \in U \exists B \in B \ni x \in B \subseteq U\).
- \(Ex: \text{open balls } B_r(x) \text{ in a metric space } (X, d) \text{ basis for the metric topology.}

- if \(T \subseteq T'\) say \(T'\) finer / \(T\) coarser.
- \(f: X \to Y\) is **continuous** if \(V \subseteq Y \text{ open} \Rightarrow f^{-1}(U) \subseteq X \text{ is open}.

  - **Closure** \(\overline{A} = \cap \text{ all closed subsets of } A\), **interior** \(\text{int}(A) = \cup \text{ all open subsets of } A\).
- \(\overline{A} = A \cup \text{ limit pts of } A\); \(\text{intersects } A\).
- **Limit Points vs. Limits of Sequences**
- **Subspace topology** on \(A \subseteq X\): \(\{ U \cap A | U \subseteq X \text{ open} \}\)
- **Product topology**: basis \(\{ \prod_{i \in I} U_i | U_i \subseteq X \text{ open}, U_i = X_i \text{ for all but finitely many } i \}\)

  - if \(U_i \ni x\), get box topology\( T\) ( finer).

  - For products of metric spaces, the uniform topology \(d_\infty(x, y) = \sup d_i(x, y) \text{ up to translation}\) is between.

- \(f = (f_i): \mathbb{Z} \to X = \prod_{i \in I} X_i\) is continuous in product top. if each \(f_i = \pi_i \circ f: \mathbb{Z} \to X_i\) is continuous.

- **Quotient topology** on \(Y = X/\sim\): \(U \subseteq Y\) open \(\iff q^{-1}(U) = \{ x \in X | [x] \subseteq U \}\) is open in \(X\).

- \(f: X \to Y\) continuous \(\iff f = \pi_q \circ f: X \to \mathbb{Z} \text{ continuous & compatible with } \sim ([x] = [x']) \Rightarrow f(x) = f(x')\).

- \(X\) is **Hausdorff** if \(\forall x, y \in \mathbb{X}; \exists U \ni x, V \ni y \text{ open st. } U \cap V = \emptyset.

- **Stronger separation ariens**: regular, normal: separate points from closed sets / closed sets from each other by disjoint open.

  - Metric spaces are normal (\(\Rightarrow\) Hausdorff).

  - Urysohn's thm: normal (or regular) space with countable basis is metrizable.

2. **Connectedness & Compactness**

- \(X\) is **connected** if \(X = U \cup V\), \(U, V\) open disjoint \(\Rightarrow\) one is \(X\) and the other is \(\emptyset\).

- \(f: X \to Y\) continuous, \(X\) connected \(\Rightarrow f(X)\) connected. (Intermediate value theorem) (connected subsets of \(\mathbb{R}\) are intervals).

- \(X\) path-connected \(\iff\) any two points of \(X\) can be joined by a path, \(f: I \to X\).

- \(X\) is **compact** if \(\{ U_i \mid i \in I \}\) open cover \(X = \bigcup_{i \in I} U_i\), \(\exists\) finite subcover \(X = \bigcup_{i \in I} U_i \ni x\).

- \(f: X \to Y\) continuous, \(X\) compact \(\Rightarrow f(X)\) compact. (Inverse image theorem).

- \(X\) compact, \(F \subseteq X\) closed \(\Rightarrow\) \(F\) compact.

- \(K \subseteq X\) Hausdorff, \(K\) compact \(\Rightarrow\) \(K\) closed.

- in \(\mathbb{R}^n\), compact \(\Rightarrow\) closed and bounded.
* Finite products of compact spaces are compact.

If $(X,d)$ is metric and compact:

- every open cover $X = \bigcup U_i$ has a Lebesgue number $\delta > 0$; diam$(A) < \delta \Rightarrow X \cap A \subset U_i$.
- every continuous function $f, (X,d) \to (Y,d_y)$ is uniformly continuous.

For metric spaces:

- compact $\iff$ limit point compact $\iff$ sequentially compact (always $\Rightarrow$).

A one-point compactification of $X$ is a compact space $Y$ s.t. $Y \setminus \{\infty\} \simeq X$.

Build: $Y = X \cup \{\infty\}$, open $= \text{finite of } X$ + complements of compact subsets of $X$.

If $X$ is locally compact ($\forall x \in X \exists \text{ neighborhood } U$ s.t. $X$ compact $C \cap U \ni x$) and Hausdorff, then $Y$ is Hausdorff and unique up to homeo.

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3. Homotopy and Fundamental Group:

- Categories, functors (language only).

- homotopy: $f, f_1 : X \to Y$ continuous; a homotopy is $F : X \times I \to Y$ continuous, $F|_{x \times \{0\}} = f_0$, $F|_{x \times \{1\}} = f_1$.

- paths $f, f_1 : I \to Y$ are path-homotopic if $\exists$ homotopy $F : I \times I \to Y$ fixing end points.

- path-homotopy classes of paths in $X$ form a groupoid for path composition $f \circ g$.

- loops based at $x_0$ $= \text{paths } x_0 \to x_0$ form fundamental group $\pi_1(X, x_0)$ (product = composition, identity = constant loop, inverse = reverse loop).

- $x_0, x_1 \in \text{same path component of } X \Rightarrow \pi_1(X, x_0) \cong \pi_1(X, x_1)$ (by attaching path $\alpha \ast f \ast \alpha^{-1}$).

- $f : (X, x_0) \to (Y, y_0)$ induces homomorphism $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$. sidewalk $((f \circ g)_* = f_* \circ g_*)$.

- Ex: $\mathbb{R}^n$, convex subsets of $\mathbb{R}^n$. $S^n$ n $\geq 2$ are simply connected ($\pi_1 = \{\{\}\}$).

- $\pi_1(S^1, b_0) \cong \mathbb{Z}$, $\pi_1(\mathbb{S}^2) \cong \mathbb{Z}^2$, $\pi_1(\mathbb{S}^1) = 0$ free group $< a, b >$.

- Application of $\pi_1(S^1) \cong \mathbb{Z}$: nontrivial $r : \mathbb{S}^1 \to S^1$ (i.e. $r$ continuous, $r|_{S^1} = \text{id}_{S^1}$).

- every continuous $f : S^2 \to S^2$ has a fixed pt $(f(x) = x)$ (Brouwer).

- $f : S^2 \to \mathbb{R}^2$ continuous $\Rightarrow \exists x \text{ st } f(x) = f(-x)$ (Borsuk-Ulam).

- Deformation retraction: $r : X \to A$ retraction ($r|_A = \text{id}_A$) st for is homotopic to $id_X$.

- Among maps that leave $A$ fixed, $i : A \to X$, $H(a, 0) = x$.

Then $\pi_1(A, a_0) \cong \pi_1(X, a_0)$ (i.e. inverse images).

$H(a, 1) \in A \forall x \in X$; $H(a, 0) = a$ for $A \subset A$.

- The same holds more generally for homotopy equivalences $X \xrightarrow{f} Y$, $g \circ f \simeq \text{id}_X$, $f \circ g \simeq \text{id}_Y$. 
• Covering spaces: \( p: E \to B, \quad \forall b \in B \exists U \ni b \text{ every } \psi \text{ covered by } p \)
\[ (p^{-1}(U)) = \text{ disjoint union of slices } V_{\psi}, \quad \text{each } p(V_{\psi}) \ni b \text{ } \text{ homeo}. \]

• Every path \( f: I \to B \) starting at \( b_0 \) has unique lift \( \tilde{f}: I \to E \) starting at \( e_0 \in f^{-1}(b_0) \). (Path) homotopy lift to (path) homotopy.

• Looking at ends points of lifts of loops in \((B, b_0)\), get lifting map \( \pi_1(B, b_0) \to \pi_1(b_0) \).

• Those loops which lift to a loop in \((E, e_0)\) form a subgroup \( H \subset \pi_1(B, b_0) \), and
\[ \pi_1(E, e_0) \cong H \subset \pi_1(B, b_0). \]

• A map \( g: (Y, y_0) \to (B, b_0) \) lifts to \( \tilde{g}: (Y, y_0) \to (E, e_0) \) iff \( g_\pi(\pi_1(Y, y_0)) \subset H \).

• Classification of covering spaces (up to equivalence) as classes of subgroups \( H \subset \pi_1(B) \) (up to conjugacy).

• Universal cover: simply-connected \( E \) (i.e., \( H \equiv \{1\} \)).

• Van Kampen: \( X = U \cup V, \quad U, V \text{ open, } U \cap V \ni x_0 \) path-connected \( \Rightarrow \)

\[ \pi_1(X, x_0) \text{ is generated by the images of } \ i_1, i_2 \text{ where } \pi_1(U) \to \pi_1(X), \pi_1(V) \to \pi_1(X) \]

• If \( \pi_1(U \cap V) = \{1\} \) then \( \pi_1(X) \) is the free product \( \pi_1(U) \ast \pi_1(V) \)

• Otherwise, quotient by smallest normal subgroup that makes \( i_1(x) \equiv i_2(x) \forall x \in \pi_1(U \cap V) \)