Recall: Def. \( x \in X \) is a limit point of \( A \subseteq X \) if, for every neighborhood \( U \) of \( x \), 
\[ U \cap (A - \{x\}) \neq \emptyset. \]

Then \( \overline{A} = \bigcap_{F \supseteq A, F \text{ closed}} F \)

Corollary: \( x \in \overline{A} \) iff \( \forall U \) neighborhood of \( x \), \( U \cap A \neq \emptyset \).

**Limits of sequences**

\( X \) top space: Then we say a sequence \( x_1, x_2, \ldots \) converges to \( x \) if, for every neighborhood \( U \) of \( x \), \( \exists N \) st. \( n \geq N \implies x_n \in U \).

(Enough to check this for a basis of neighborhoods of \( x \), i.e., a family of neighborhoods s.t. every neighborhood of \( x \) contains one of them.)

E.g. in a metric space, balls \( B_r(x), r > 0 \), or even the balls \( B_{1/n}(x) \).

\( \implies \) taking base, recover usual notion:
\( x_n \to x \) if \( \forall r > 0 \exists N \) s.t. \( n \geq N \implies x_n \in B_r(x) \).

Fact: if \( \exists \) sequence \( x_n \) in \( A \) with \( x_n \to x \) \( \forall n \) and \( x_n \to x \), then \( x \) is a limit point of \( A \).

(in fact: \( \forall U \) neighborhood of \( x \), \( U \cap (A - \{x\}) \exists x_n \) for all sufficiently large \( n \) !)

Conversely, in a metric space, if \( x \) is a limit point of \( A \subseteq X \) then \( \forall n > 0 \exists x_n \in B_{1/n}(x) \cap A \) with \( x_n \to x \).

Hence \( x \) is the limit of a sequence \( \{x_n\} \) in \( A \) with \( x_n \to x \).

This holds more generally in spaces where points have countable bases of neighborhoods \( U, U_2, \ldots \) (i.e., \( \forall x \exists \text{basis } U_1, U_2, \ldots \text{ s.t. } \forall x \exists U_x \cap U, x \in U_x \subseteq U \)), but not in arbitrary topological spaces!

**Example:** Let \( X = \mathbb{R} \) with topology \( \mathcal{T} = \{ U \subseteq \mathbb{R} / U = \emptyset \text{ or } \mathbb{R} - U \text{ is countable} \} \).
(check this satisfies the axioms). Let \( A = (0, 1) \). Then \( \overline{A} = \mathbb{R} \)

(Indeed: closed \( \implies \) countable or all \( \mathbb{R} \), so smallest closed set \( \supseteq (0, 1) \) is \( \mathbb{R} \).)

Hence \( 2 \) is a limit point of \( A \)!

But there is no sequence \( a_n \in A \) s.t. \( a_n \to 2 \), since the complement of any sequence in \( A \) is open, hence a neighborhood of 2 containing no \( a_n \)'s.

(in fact for a seq. to converge in this topology it must be constant after finitely many terms).
Hausdorff spaces:

Recall: in a metric space, a sequence converges to at most one limit.

This is not true in an arbitrary topological space!

Ex: \( X = \mathbb{R} \) with finite complement topology (open = \( \emptyset \) and \( \mathbb{R} - \{ \text{finite sets} \} \)).

Let \( a_1, a_2, \ldots \) be a sequence in \( X \) with all \( a_i \) distinct.

Then \( \forall x \in X \), every neighborhood \( U \ni x \) has finite complement, hence
contains all but finitely many of the \( a_i \), hence \( \exists N \) s.t. \( a_n \notin U \) \( \forall n > N \).

Thus the sequence converges to every point of \( X \)!

To avoid such pathological behavior:

Def: A top-space is **Hausdorff** if \( \forall x_1 \neq x_2 \in X \), \( \exists \) neighborhoods \( U_1 \ni x_1, U_2 \ni x_2 \)

s.t. \( U_1 \cap U_2 = \emptyset \).

Ex: 1) any metric space is Hausdorff:

\[
\begin{align*}
\text{given } x_1 \neq x_2, & \text{ choose } 0 < \varepsilon < \frac{1}{2} d(x_1, x_2) \\
\text{then } U_i = B_{\varepsilon}(x_i) & \text{ disjoint neighborhoods of } x_i.
\end{align*}
\]

2) the finite complement topology on \( \mathbb{R} \) is not Hausdorff, since any two non-empty open sets intersect (in infinitely many points).

3) the discrete topology is always Hausdorff (\( U_i = \{x_i\} \) disjoint neighborhoods of \( x_i \))

Thm: if \( X \) is Hausdorff then every sequence in \( X \) converges to at most one limit.

Proof: assume \( x_1, x_2, \ldots \) converge to \( x \in X \), and let \( y \neq x \).

Choose \( U_x \ni x \), \( U_y \ni y \) disjoint neighborhoods.

Since \( x_n \to x \), \( \exists N \) s.t. \( \forall n > N \) \( x_n \notin U_y \). Hence \( x_n \notin U_y \) for \( n > N \),
so the sequence doesn’t converge to \( y \). \( \Box \)

Remark: there are several flavors of separation axioms, beside the notion of Hausdorff-ness:

- say \( X \) is \( T_0 \) if \( \forall x \neq y \), there exists an open set containing one but not the other
- say \( X \) is \( T_1 \) if \( \forall x \neq y \), \( \exists \) neighborhood \( U \ni y \) which doesn’t contain \( x \).

(\( \Leftrightarrow \forall x, \{x\} \text{ is closed} \)) (indeed; consider \( \mathbb{R} - \{x\} \text{ vs. } U \ni x \).)

- say \( X \) is \( T_2 \) if \( \forall x \neq y \), \( \exists U_x, U_y \ni x, y \) open and disjoint.

(\( \Leftrightarrow \forall x, \text{ \{closed\} } \Rightarrow \text{ \{compact\} } \)) (indeed; consider \( \text{compact-ness vs. separation} \)).
1. $T_2$: Hausdorff. $\forall x \neq y$. $\exists$ neighborhoods $U \ni x$, $V \ni y$ s.t. $U \cap V = \emptyset$.

2. $T_3$: regular: $T_1 + \forall x \in X$, $\forall A$ closed s.t. $x \notin A$, $\exists$ open $U \ni x$, $V \supseteq A$, $U \cap V = \emptyset$.

3. $T_4$: normal: $T_1 + \forall A, B$ $\subseteq X$ closed & disjoint, $\exists$ open $U \supseteq A$, $V \supseteq B$, $U \cap V = \emptyset$.

Ex: $R$ with the finite complement topology is $T_1$ ($\mathbb{R} - \{x\}$ open $\forall x$) but not Hausdorff (as seen above).

Ex: $R_e$ is normal; $R_e < R_{e}$ is regular but not normal. (Munkres end of §31).

The motivation for studying normal & regular spaces comes from the question of metrizability, i.e. which topologies are actually metric space topologies.

Then: || Every metric space is normal

PF: let $A, B \subseteq X$ closed & disjoint.

- $\forall a \in A \exists \varepsilon_a > 0$ s.t. $B(a, \varepsilon_a) \subseteq X - B$.
- $\forall b \in B \exists \varepsilon_b > 0$ s.t. $B(b, \varepsilon_b) \subseteq X - A$.

Observe: $d(a, b) \geq \max(\frac{\varepsilon_a}{2}, \frac{\varepsilon_b}{2}) = \frac{\varepsilon_a + \varepsilon_b}{2}$ $\forall a \in A \forall b \in B$, hence $B(a, \frac{\varepsilon_a}{2}) \cap B(b, \frac{\varepsilon_b}{2}) = \emptyset$.

This implies: $U = \bigcup_{a \in A} B(a, \frac{\varepsilon_a}{2})$ and $V = \bigcup_{b \in B} B(b, \frac{\varepsilon_b}{2})$ are disjoint ($\&$ clearly open, contain $A \& B$). □

Urysohn metrization theorem: || Every regular space with a countable basis is metrizable.

(i.e. $\exists$ metric inducing the topology).

---

Topologies on products (§13)

Given an index set $I$, and topological spaces $X_i \ni i \in I$, consider the product set $X = \prod_{i \in I} X_i = \{ (a_i)_{i \in I} \mid a_i \in X_i \; \forall i \in I \}$

Natural topology on $X$?

First idea: the box topology

Def: the box topology on $\prod_{i \in I} X_i$ has basis $\{ \prod_{i \in I} U_i \mid U_i \subseteq X_i \text{ open} \forall i \}$

This is a basis: box $\cap$ box = box, since $(\prod_i U_i) \cap (\prod_i V_i) = \prod_i (U_i \cap V_i))$
This is a natural definition, but has unexpected properties.

Example: consider the diagonal map \( \Delta : \mathbb{R} \to \mathbb{R}^\infty \Rightarrow \mathbb{R}^\omega \) where \( \Delta(x) = (x, x, x, \ldots) \)

For finite products, with product topology, \( \Delta : \mathbb{R} \to \mathbb{R}^n \) is continuous (in fact, an embedding).

But, giving \( \mathbb{R}^\infty \) the box topology, \( \Delta \) is not continuous!!

Indeed, let \( U = (-1, 1) \times (-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{3}, \frac{1}{3}) \times \cdots \) open in box topology.

\[ \Delta^{-1}(U) = \bigcap_{n=1}^\infty (-\frac{1}{n}, \frac{1}{n}) = \{0\} \] not open in \( \mathbb{R}^n \).

Better: the product topology

**Define** the product topology on \( X = \prod X_i \) has basis:

\[ \{ \prod U_i \mid U_i \subset X_i \text{ open, and } U_i = X_i \text{ for all but finitely many } i \} \]

(check this is a basis!)

(This is the same as the box topology if \( I \) is finite; for infinite \( I \), this is coarser than the box topology)

Unless otherwise specified, the product topology is the one we'll use on \( \prod X_i \).

**Theorem:** \( f : \mathbb{Z} \to X = \prod X_i \) is continuous \( \iff \) each component \( f_i : \mathbb{Z} \to X_i \) is continuous.

**Proof:**

- **If** \( f \) is continuous. The component maps are \( f_i = p_i \circ f \) where \( p_i : X \to X_i \) is the projection to the \( i \)th factor.
  - \( p_i \) is continuous since \( U \subset X_i \text{ open} \Rightarrow p_i^{-1}(U) = \text{product of } \{X_j \text{ for } j \neq i \text{ open} \} \)
  - Hence \( f_i = p_i \circ f \) is continuous (composition of 2 continuous functions)

- **Conversely,** assume all \( f_i \) are continuous, and consider basis element \( \prod U_i \subset X \) where \( U_i = X_i \text{ for all but finitely many } i \).
  - Then \( f^{-1}(\prod U_i) = \{ z \in \mathbb{Z} \mid (f_i(z)), i \in I \in \prod U_i \} = \bigcap_{i \in I} f_i^{-1}(U_i) \)
  - Each \( f_i^{-1}(U_i) \subset \mathbb{Z} \) is open, and all but finitely many are \( f_i(X_i) = \mathbb{Z} \), so can be omitted from the intersection.

So \( f^{-1}(\prod U_i) \) is the intersection of finitely many open sets in \( \mathbb{Z} \), hence open. \( \Box \)