Def. A function \( f: X \to Y \) between topological spaces is continuous if 
\( \forall U \subseteq Y \text{ open, } f^{-1}(U) \subseteq X \text{ is open.} \)

(Various examples seen last time)

* It suffices to check continuity on elements of a basis!

Prop. \( f: X \to Y \) is continuous iff \( f^{-1}(B) \subseteq X \) is open for all \( B \) in a basis for the topology on \( Y \).

Prof. \( f^{-1}(B) \) open \( \forall B \subseteq \text{basis} \) is obviously necessary for continuity of \( f \),

since every basis element is open in \( Y \).

* Every open \( U \subseteq Y \) can be written as \( U = \bigcup_{i \in I} B_i \), \( B_i \subseteq \text{basis} \). Since

\[ f^{-1}(U) = \bigcup_{i \in I} f^{-1}(B_i), \]

if \( f^{-1}(B_i) \) are all open in \( X \) then so is \( f^{-1}(U) \). \( \Box \)

Ex: \( X \) any top. space, \( Y \) metric space, then to check continuity of \( f: X \to Y \) it is

enough to check that \( f^{-1}(B_r(y)) \) is open \( \forall y \in Y, \forall r > 0. \)

Properties of continuous functions: for any topological space:

Thm: 1) constant functions \( f: X \to Y, f(x) = y_0, \forall x \in X \) for some fixed \( y_0 \in Y \)

are continuous.

2) if \( A \subseteq X \) is given the subspace topology, then the inclusion \( i: A \to X \)

is continuous.

3) if \( f: X \to Y \) and \( g: Y \to Z \) are continuous, then \( g \circ f: X \to Z \) is continuous.

4) if \( X = \bigcup_{\alpha} U_\alpha \) with \( U_\alpha \) open ("open cover of \( X \)) , and

\( f: X \to Y \) a function such that \( f|_{U_\alpha}: U_\alpha \to Y \) is continuous for all \( \alpha \),

then \( f \) is continuous with subspace topology.

PF: 1) if \( f \) is a constant function then \( \forall U \subseteq Y \), \( f^{-1}(U) = \left\{ x \mid y_0 \in U \right\} \) always open.

2) \( \forall U \subseteq X \text{ open, } f^{-1}(U) = U \cap A \text{ is open in } A. \)

3) \( \forall U \subseteq Z, (g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U)) \) \( \left( g(f(x)) \in U \iff f(x) \in g^{-1}(U) \iff x \in f^{-1}(g^{-1}(U)) \right) \)

\( U \text{ open in } Z \Rightarrow g^{-1}(U) \text{ open in } Y \Rightarrow f^{-1}(g^{-1}(U)) \text{ open in } X. \)
4) \( \forall V \subseteq Y \text{ open}, \quad (f(U_\alpha))^{-1}(V) = f^{-1}(V) \cap U_\alpha, \) so \( f^{-1}(V) = \bigcup_\alpha (f(U_\alpha))^{-1}(V). \)

\( (f(U_\alpha))^{-1}(V) \) is open in \( U_\alpha, \) so it's the intersection of \( U_\alpha \) with an open subset of \( X, \) hence (since \( U_\alpha \) also open) it's an open subset of \( X. \)

\( f^{-1}(V) \) is therefore a union of open sets in \( X, \) hence open. \( \Box \)

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**Homeomorphisms:** two topological space \( X \) and \( Y \) are "homeomorphic" if they are topologically the same - namely, if there exists a bijection \( f: X \to Y \) s.t. \( U \) is open \( \iff f(U) \) is open.

**Def:** A bijection \( f: X \to Y \) is a **homeomorphism** if \( f \) and \( f^{-1} \) are both continuous.

- Say \( X \) and \( Y \) are **homeomorphic** if there exists a homeomorphism between them.

**Ex:** we've seen that a continuous bijection need not be a homeomorphism.

- \( f = \text{id}: \mathbb{R} \to \mathbb{R} \) is continuous \( (a,b) \subset \mathbb{R} \text{ open} \) but \( f' \) isn't \( \left( [a,b] \subset \mathbb{R}, \text{Id} \not\text{not open} \right). \)

**Ex:** \( X = \{0\} \cup \{\frac{1}{2}, \frac{1}{3}, \ldots\} \) with topology induced by metric of \( \mathbb{R} \)

\( N = \{0, 1, 2, 3, \ldots\} \) with discrete topology

Define \( f: N \to X \) by \( f(0) = 0, f(n) = \frac{1}{n} \) for \( n \geq 1. \)

This is continuous (in fact any function from discrete top. is continuous since all subsets are open) and bijective, but not a homeomorphism

(\( \{0\} \subset X \) is not open, since any open ball around 0 contain \( \frac{1}{n} \) for large \( n \)).

\( f \) is a homeomorphism since every subset of \( N \) is open whereas not true for \( X. \)

- A metric space is **bounded** if \( \sup \{d(x, y) \mid x, y \in X \} < \infty. \)

This is not a topological property! For example:

\( f: (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}, \ f(x) = \tan x \)

This is a continuous bijection, and \( f^{-1} = \arctan \) is continuous as well, so \( (-\frac{\pi}{2}, \frac{\pi}{2}) \) is homeomorphic to \( \mathbb{R}. \) (or in fact, any open interval).

**Def:** \( f: Y \to X \) continuous injective map, then \( f \) is an **embedding** if the map \( Y \to f(Y) \) is a homeomorphism (with subspace topology on \( f(Y) \subset X). \)

**Ex:** \( f: N \to \mathbb{R}, \ f(0) = 0, f(n) = \frac{1}{n} \) for \( n \geq 1 \) is not an embedding.

- \( Y \subset X \) with subspace topology, the inclusion \( i: Y \to X \) is an embedding.
Beware: in differential topology the notion of embedding is different, because one considers spaces not just topologically but with a smooth structure.

\[ \mathbb{R} \rightarrow \mathbb{R}^2 \quad \alpha \mapsto (\alpha, \alpha^2) \quad \text{is a topological embedding but not a smooth embedding.} \]

Closed sets & limit points (Munkres §17)

Recall: a subset \( A \) of a topological space \( X \) is closed if \( X \setminus A \) is open.

(sets can be both closed & open, e.g. \( \emptyset \) and \( X \), or neither)

**Def:** \( A \subset X \) any subset

1) the closure of \( A \), \( \bar{A} \) = smallest closed set containing \( A \)

\[ \bar{A} = \bigcap \{ F \mid F \supseteq A, F \text{ closed} \} \]

(closed since it's an intersection of closed sets)

2) the interior of \( A \), \( \text{int}(A) \) = largest open set contained in \( A \)

\[ \text{int}(A) = \bigcup_{U \subset A, U \text{ open}} U \]

3) the boundary of \( A \) is \( \partial A = \bar{A} \setminus \text{int}(A) \) (or \( \text{bd}(A) \))

**Ex:** \( A = [0,1) \subset \mathbb{R} \Rightarrow \bar{A} = [0,1], \text{int}(A) = (0,1), \partial A = \{0,1\} \)

**Rmk:** \( A \) is closed iff \( \bar{A} = A \), open iff \( \text{int}(A) = A \).

**Def:** say \( A \) is dense if \( \bar{A} = X \).

**Ex:** \( Q \subset \mathbb{R} \) is dense in \( \mathbb{R} \).

Indeed, assume not, then \( \exists x \in \mathbb{R} \setminus Q \) which is open

\[ \Rightarrow \exists a < b \text{ s.t. } x \in (a, b) \subset \mathbb{R} \setminus Q \subset \mathbb{R} \setminus Q. \text{ But } \exists \text{rationals in } (a, b). \]

**Def:** \( U \subset X \) is a neighborhood of \( x \in X \) if \( x \in U \) and \( U \) is open.

**Def:** \( x \in X \) is a limit point of \( A \subset X \) if, for every neighborhood \( U \) of \( x \), \( U \cap (A \setminus \{x\}) \neq \emptyset \).

\( \times \) is not a limit point

**Ex:** in \( \mathbb{R} \setminus \{0\} \), \( 1 \) is a limit point of \( (0,1) \) and of \( [0,1] \).

\( 1 \) is not a limit point of \( \{\frac{1}{n}, n \geq 1\} \cup \{0\} \), but \( 0 \) is.

\[ \mathbb{R} \setminus \{0\} \]

\[ \{\frac{1}{n}, n \geq 1\} \cup \{0\} \]
Then, \( \overline{A} = A \cup \{ \text{limit points of } A \} \).

**Pf:**
1. Suppose \( x \notin A \) and \( x \) isn't a limit point. Then \( \exists U \) neighborhood of \( x \) such that \( U \cap A = \emptyset \).
   Hence \( A \cap X - U \), which is closed \( \Rightarrow \overline{A} = \bigcap \{ \text{closed sets containing } A \} \subseteq X - U \).
   So \( U \cap \overline{A} = \emptyset \), hence \( x \notin \overline{A} \).
2. Conversely, suppose \( x \in \overline{A} \). Then \( U = X - \overline{A} \) is an open neighborhood of \( x \) disjoint from \( A \), so \( x \) is not a limit point of \( A \) (nor in \( A \)). \( \square \).

**Corollary:** \( x \in \overline{A} \iff \forall U \) neighborhood of \( x \), \( U \cap A \neq \emptyset \). (Proof: consider separately cases \( x \in A \), \( x \notin A \).)