Recall: a basis \( B \) generates a topology \( \tau = \{ U \times V \mid V \in \mathcal{V}, \exists B \in B : x \in B \subseteq U \} \)

\( (\tau = \text{coarsest topology for which all \( x \in B \) are open). (eg balls in a metric space)} \)

Example: the subspace topology (Munkres §16)

- \( X \) topological space, \( A \subseteq X \).
- If \( X \) metric space then \( A \) inherits a metric by restriction and then \( B^A_r(p) = B^X_r(p) \cap A \)
  
  \( \forall p \in A, r > 0 \)

This motivates the following definition, when \( X \) is a topological space:

\[ \text{Def:} \quad \text{the subspace topology on } A \text{ is defined by } \tau_A = \{ U \cap A : U \in \tau_X \} \]

Easy to check:

1. This is a topology on \( A \).
2. If \( B \) is a basis for \( \tau_X \), then \( B_A = \{ B \cap A : B \in B \} \) is a basis for \( \tau_A \).

Ex: in \( [0,1] \subset \mathbb{R} \) with subspace topology, \( (-1,1) \cap [0,1] = [0,1] \) is open.

- Subspace topology on \( \mathbb{R} = \{ \text{x-axis} \} \subset \mathbb{R}^2 \), standard top is standard topology.

Example: the product topology on \( X \times Y \), given topologies \( \tau \) on \( X \) and \( \tau' \) on \( Y \\)

\( (\mathfrak{S}15) \)

is generated by the basis \( B = \{ U \times V : U \in \tau, V \in \tau' \} \).

Claim: \( B \) is a basis.

Proof: 1) \( X \times Y \in B \), so the union gives \( X \times Y \in \bigcup_{\tau \in \tau} \bigcup_{\tau' \in \tau'} \).

2) If \( U_1 \times V_1, U_2 \times V_2 \in B \), then \( (U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2) \in B \)

(recall: would be enough: intersection of two elements of \( B \) is a union of elements of \( B \).

Remark: not every open subset in the product topology is of the form \( U \times V \), \( U \in \tau, V \in \tau' \).

eg. look at \( (U_1 \times V_1) \cup (U_2 \times V_2) \) in above example!

There's in fact a slightly more efficient basis for the product topology.
Claim: If \( B, B' \) are bases for \( T, T' \) respectively, then \( D = \{ B \times B' \mid B \in B, B' \in B' \} \) is a basis for the product topology.

**Proof:** \( D = \{ U \times V \mid U \in T, V \in T' \} \), but \( U \in T \Rightarrow U = \bigcup_{i \in I} B_i, B_i \in B \) then \( U \times V = \bigcup_{i \in I} U_x B_i \times B_j \) so \( D \) generates the same topology.

\[ U \times V = \bigcup_{i \in I} U_x B_i \times B_j \]

So \( D \) generates the same topology.

**Example:** \( R \times R \), the product topology coincides with the standard topology of \( R^2 \).

Indeed: a basis for product topology consists of rectangles \( (a_1, b_1) \times (a_2, b_2) \) which are all open in the standard topology.

Conversely, check that the elements of a basis for the standard topology are open for the product topology. This can be done for balls of the Euclidean metric:

\[ \text{3D Euclidean ball} \]

but even more easily for the balls of the metric \( d_{\infty} \) which defines the same topology as the Euclidean one (see HW1): balls for \( d_{\infty} \) are rectangles \( (x-r, x+r) \times (y-r, y+r) \) so open in the product topology.

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**Example (Exercise 514):** the order topology (less fundamentally important than subspace & products, but still a good source of examples).

Let \( X \) be a set with a total ordering:

- \( \forall a, b \in X \), exactly one of \( a < b, b < a, a = b \) holds.
- \( \forall a, b, c \in X \), \( a < b \) and \( b < c \) \( \Rightarrow \) \( a < c \).

The order topology on \( X \) is generated by the basis consisting of open intervals \( (a, b) = \{ x \in X \mid a < x < b \} \), and possibly half-open intervals if \( X \) has a smallest element \( a_0 \), \( [a_0, b) \) (check this is a basis).

On \( R \) (with usual order) this generates the standard topology.

**Example:** let \( I = [0, 1] \), equip \( I \times I \) with the dictionary order \( a \leq b \iff (a < a') \text{ or } (a = a' \text{ and } b < b') \)

Then basis elements look like:

Not open in the standard topology!
Open sets in the standard topology are also not always open in order topology!

**Example:** Consider the Euclidean metric:

- A ball centered at \((\frac{1}{2}, 0)\) is not open in the order topology, because any open interval containing \((\frac{1}{2}, 0)\) must be \((a_0, a_0 + l)\) with \(a_0 + l < \frac{1}{2} \Rightarrow a < \frac{1}{2}\) and \(a' \geq \frac{1}{2}\) so contains a whole vertical interval \(x \times I\) for \(a < x < a'\).

So this topology is not comparable to the standard one (neither is contained in the other).

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**Continuous functions**

**Definition:** A function \(f : X \to Y\) between topological spaces is **continuous** if \(f^{-1}(U) \subseteq X\) is open.

We've seen that, for metric spaces, this agrees with the \(\varepsilon-\delta\) definition of continuity.

\[ \forall x \in X, \forall \varepsilon > 0 \exists \delta > 0 \text{ s.t. } B_{\delta}(x) \subseteq f^{-1}(B_\varepsilon(f(x))). \quad (\text{cf. lecture 5}). \]

**Example:** \(f : \mathbb{R} \to \mathbb{R}\) defined by \(f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}\)

is discontinuous (for the standard topology). What goes wrong with \(f^{-1}(\text{open sets})\)?

**Answer:** e.g. \(f^{-1}((-1, 1]) = [0, \infty)\) not open.

On the other hand, \(f\) is continuous if we equip \(\mathbb{R}\) with the **lower limit topology**!

Indeed, \(f^{-1}\) of any subset of \(\mathbb{R}\) is one of \(\emptyset, (-\infty, 0], [0, \infty), IR\) — all of which are open in the lower limit topology! So \(f : \mathbb{R} \to \mathbb{R}_L\) is continuous.

**Example:** \(f : \mathbb{R} \to \mathbb{R}_L\) identity function \(f(x) = x\).

This isn't continuous, since \(f^{-1}((0, 1)) = (0, 1)\) isn't open in the standard topology.

However, the identity function \(\mathbb{R}_L \to \mathbb{R}\) is continuous, since \(\mathbb{R}_L\) has a finer topology.

\((U \subseteq \mathbb{R}\text{ open } \Rightarrow f^{-1}(U) = U \text{ open in } \mathbb{R}_L)\).

**Example:** \(X, Y\) top spaces, equip \(X \times Y\) with the **product topology**.

Let \(p_1 : X \times Y \to X\) be first projection. Then \(V \subseteq X\text{ open } \Rightarrow p_1^{-1}(V) = U \times Y \text{ is open in } X \times Y\), hence \(p_1\) is continuous.