The Borsuk-Ulam Theorem for $\mathbb{S}^2$: ($\S$57)

If $f: \mathbb{S}^2 \to \mathbb{R}^2$ is continuous, then there exists a pair of antipodal points $x, -x \in \mathbb{S}^2$ such that $f(x) = f(-x)$.

Similarly for $f: \mathbb{S}^n \to \mathbb{R}^n$ in general; case $n=1$ follows from intermediate value theorem, case $n \geq 3$ requires more algebraic topology.

Let's first start with simpler steps.

**Def**: if $x \in \mathbb{S}^n$, its antipode is $-x \in \mathbb{S}^n$. A map $h: \mathbb{S}^n \to \mathbb{S}^m$ is antipode-preserving if it maps antipode to antipode: $h(-x) = -h(x) \forall x \in \mathbb{S}^n$.

**Ex**: rotation of $\mathbb{S}^1$ by angle $\theta$ is antipode-preserving. Viewing $\mathbb{S}^1$ as unit complex numbers, $r_{\theta}(z) = e^{i\theta}z$, and $r_{\theta}(-z) = e^{i\theta}(-z) = -r_{\theta}(z)$.

**Thm**: if $h: \mathbb{S}^n \to \mathbb{S}^1$ is continuous and antipode-preserving then it is not nullhomotopic.

**Pf**: (By hand — see Nandra for a proof that generalizes to higher dim's)

We show $h_*: \pi_1(\mathbb{S}^1) \to \pi_1(\mathbb{S}^n)$ (i.e. $\mathbb{Z} \to \mathbb{Z}$) is a nontrivial homomorphism.

Let $\alpha: \mathbb{S}^1 \to \mathbb{S}^n$ be the antipodal map, and let $f: I \to \mathbb{S}^1$ be $s \mapsto (\cos \pi s, \sin \pi s)$

path $b_0 = (1,0) \to -b_0$ going halfway around $\mathbb{S}^1$, so $g = f \circ (\alpha \circ f)$ is a loop going once around.

Then $h \circ f: I \to \mathbb{S}^1$ is a path from $h(b_0) = h(-b_0) = -h(b_0)$, and $h \circ \alpha = \alpha \circ h \Rightarrow h \circ (\alpha \circ f) = \alpha \circ (h \circ f)$ path from $-h(b_0)$ to $h(b_0)$,

& the composition $(h \circ f) \times (\alpha \circ (h \circ f)) = h \circ (f \times (\alpha \circ f)) = h \circ g$ is then a loop representing

the homotopy class $h_*([g])$, which we wish to show is nontrivial.

Now, consider path-lifting to the covering space $p: \mathbb{R} \to \mathbb{S}^1$ when lifting map gives $t \mapsto (\cos 2\pi t, \sin 2\pi t)$, $\pi_1(\mathbb{S}^1) = \mathbb{Z}$.
Ride a lift to of h(s), the lift k of h of starting at to is a path in R which ends at a point of p^1(-h(s)) = to + n + 1/2 , say to + n + 1/2 for some n ∈ Z.

The lift of αo(h(s)) starting at to + n + 1/2 is then l: s → k(s) + n + 1/2 ,
which is a path from to + n + 1/2 to to + 2n + 1.

Hence the lift of hog = (h(s)h(s)h(s)) starting at to is k + 1, which is a path in R from to to to + 2n + 1. We conclude that under η̂(S) = Z,
\[
[hog] \mapsto 2n + 1.
\]

Since 2n + 1 ≠ 0, this implies that h(S) is non-trivial, and hence h isn’t nullhomotopic.

**Corollary:** There is no continuous antipode-preserving map g: S^2 → S^1.

**Proof:** Suppose g is continuous and antipode-preserving, & consider the equator S^1 ⊂ S^2.

Then g|S^1: S^1 → S^1 is an antipode-preserving map, hence not nullhomotopic by the previous theorem, however it extends to B^2 (northern hemisphere), contradiction.

\[g|B^2: B^2 → S^1\]

**Corollary (Borsuk-Ulam than for S^2):** If f: S^2 → R^2 is continuous then ∃ x ∈ S^2 s.t. f(x) = f(-x).

**Proof:** Assume not, then f(x) ≠ f(-x) ∀x, so
\[g(x) = \frac{f(x) - f(-x)}{|f(x) - f(-x)|} : S^2 → S^1 \text{ is continuous and}
\]
\[g(-x) = -g(x) \forall x, \text{ contradiction.}\]

**Corollary:** An open set in R^2 cannot be homeomorphic to an open set in R^n for n > 3.

(For R^2 vs. R^n vs. this is much easier: removing a point from (a,b) disconnects it.)

**Proof:** Assume U ⊂ R^n open and f: U → V ⊂ R^2 homeo.

Then ∃ Br(x) ⊂ U for some small r > 0, which is homeomorphic to B^2 ⊂ B^2 ⊂ S^2.

So by restriction we get a continuous, injective map f|S^2: S^2 → R^2.

This contradicts Borsuk-Ulam.

**A fun application:**

Given a bounded polygonal (or “nice enough” - measurable suffice :) region A ⊂ R^2, ∃ straight lines in R^2 that bisects it into equal area.

This is easy by intermediate value theorem:

If continuous, ∃ c s.t. f(c) = 1/2 Area(A)
Using Borsuk-Ulam, one shows:

**Theorem:** Given 2 bounded (polygonal) regions $A_1, A_2 \subset \mathbb{R}^2$, there exists a straight line in $\mathbb{R}^2$ that simultaneously bisects each of them into equal areas.

**Proof:** Place $A_1, A_2$ in the plane $\mathbb{R}^2 \times \{1\} \subset \mathbb{R}^3$.

Given a unit vector $u \in S^2$, let $P \subset \mathbb{R}^3$ be the plane through origin with normal vector $u$. $P$ divides $\mathbb{R}^2$ into two half-spaces, and $\mathbb{R}^2 \times \{1\}$ into two half-planes (usually).

Let $f_i(u) =$ area of the part of $A_i$ that lies on the side of $u$.

![Diagram showing the concept](image)

Notes: $f_i(u) + f_i(-u) = \text{area}(A_i)$

Now, $F(u) = (f_1(u), f_2(u))$ is a continuous map $S^2 \to \mathbb{R}^2$.

So $\exists u \in S^2$ st. $F(u) = F(-u) \implies f_i(u) = f_i(-u) = \frac{1}{2} \text{area}(A_i)$. 

This generalizes to: given $n$ bounded measurable sets in $\mathbb{R}^n$, there exists a hyperplane which bisects them all evenly. In $\mathbb{R}^3$ this is called the "ham sandwich theorem."