We'll now explore neat topological applications of \( \pi_1(S^1) = \mathbb{Z} \). Consider two-dimensional analogues of the intermediate value theorem, specifically:

1. every continuous map \( f: I \to I \) has a fixed point (\( \exists x \in I \) st. \( f(x) = x \))
   
   \[ \text{Proof: let } g(x) = f(x) - x, \text{ apply intermediate value theorem to } g \ (g(0) \geq 0, g(1) \leq 0). \]

2. if \( f: S^1 \to \mathbb{R} \) is continuous, then \( \exists x \in S^1 \) st. \( f(x) = f(-x) \).
   
   (This was an HW4, proof considers \( g(x) = f(x) - f(-x) \) & connectedness of \( S^1 \)).

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1. **The Brouwer fixed point theorem:**

   Let \( B^n \) denote the closed ball of radius \( 1 \) in \( \mathbb{R}^n \), with boundary the unit sphere \( S^{n-1} \).

   Recall that if \( A \subset X \), a retraction \( r: X \to A \) is a continuous map s.t. \( r(a) = a \forall a \in A \).

   Then: there is no retraction of \( B^2 \) onto \( S^1 \).

   \[ \text{Proof: if } r: B^2 \to S^1 \text{ is a retraction, then given any loop } f \text{ in } S^1, f \text{ is also a loop in } B^2, \text{ which is convex } \subset \mathbb{R}^2, \text{ so } \exists \text{ straight line homotopy } F: I \times I \to B^2 \text{ from } f \text{ to constant loop. Then } r \\
   \text{of } F: I \times I \to S^1 \text{ gives a homotopy in } S^1 \text{ from } f \text{ to constant loop. This contradicts } \pi_1(S^1) \\neq 1. \]

   (see also HW).

   [with more alg. top., similarly \( \not\exists \text{ retraction } B^n \to S^{n-1} \forall n \)].

2. **Brouwer fixed point theorem:**

   If \( f: B^2 \to B^2 \) is continuous, then \( \exists x \in B^2 \) st. \( f(x) = x \).

   [with more alg. top., the same holds for continuous maps \( B^n \to B^n \forall n \)].

   **Proof #1** ("by hand"): assume \( f: B^2 \to B^2 \) continuous, \( f(x) \neq x \ \forall x \in B^2 \).

   Then define \( h: B^2 \to S^1 \) by mapping each \( p \in B^2 \) to the point where the ray from \( f(p) \) to \( p \) hits \( \partial B^2 = S^1 \).

   (Formula: \( h(p) = p + t(p - f(p)) \) where \( t > 0 \) s.t. \( \| h(p) \|^2 = 1 \). can solve by quadratic formula, so \( t \) does depend continuously on \( p \).)

   This gives a continuous map \( h: B^2 \to S^1 \), moreover if \( p \in S^1 \) then \( h(p) = p \), so we get a retraction \( B^2 \to S^1 \). Contradiction.

   \[ \square \]

   We'll give a more conceptual version of this argument, after some useful lemmas...
Lemma: Let \( h: S^1 \to X \) be continuous, then the following are equivalent:

1. \( h \) is nullhomotopic
2. \( h \) extends to a continuous map \( k: B^2 \to X \) \((k|_{\partial B^2} = h)\).
3. \( h_*: \pi_1(S^1) \to \pi_1(X) \) is the trivial homomorphism.

Proof:

1) \(\Rightarrow\) 2): Let \( H: S^1 \times I \to X \) be a homotopy between \( h \) and a constant map.
Define a map \( \Pi: S^1 \times I \to B^2 \) by \( \Pi(x,t) = (1-t)x \):

\[
\begin{array}{c}
\text{Disk} \\
\longrightarrow \\
\text{Circle}
\end{array}
\]

Can check \( \Pi \) is a quotient map, collapsing \( S^1 \times \{1\} \) to a point (the origin), and a homeomorphism on \( S^1 \times [0,1) \to B^2 \setminus \{0\} \).

Since \( H: S^1 \times I \to X \) is constant on \( S^1 \times \{1\} \), it induces a continuous map \( (S^1 \times I)/\sim \to X \), i.e. \( \exists k: B^2 \to X \) s.t. \( H = k \circ \Pi \). \( S^1 \times I \to B^2 \to X \)

Moreover, \( \Pi \) maps \( S^1 \times \{0\} \) to \( S^1 \times \{0\} \), so \( k|_{S^1 \times \{0\}} \) agrees with \( h|_{S^1 \times \{0\}} = h \).

2) \(\Rightarrow\) 3): if \( h|_{S^1} \) is null, then we may write \( h = k \circ i \) where \( i: S^1 \to B^2 \) is the inclusion.

By functoriality of \( \pi_1 \), \( h_* = k_* \circ i_* : \pi_1(S^1) \to \pi_1(B^2) \to \pi_1(X) \)

But \( \pi_1(B^2) = \{1\} \), so \( k_* \) is trivial and so is \( h_* \).

3) \(\Rightarrow\) 1): assume \( h_*: \pi_1(S^1, b_0) \to \pi_1(X, x_0) \) is trivial.

Let \( f: I \to S^1 \) be the loop \( f(s) = (\cos 2\pi s, \sin 2\pi s) \), representing the generator of \( \pi_1(S^1) \).

This is also a quotient map \( [0,1]/\sim \) \( S^1 \).

Then \( g = h \circ f: I \to X \) is a loop in \( X \), representing \( h_*([f]) = 1 \).

Thus \( \exists \) path homotopy \( G: I \times I \to X \) from \( g \) to the constant path at \( x_0 \).

Now, \( F: I \times I \to S^1 \times I \) is a quotient map identifying \((0,t) \sim (1,t)\),

\( (s,t) \to (f(s),t) \)

and since \( G(\cdot, t) = G(1,t) = x_0 \) \( \forall t \), \( G: I \times I \to X \) induce a continuous map

\( H: S^1 \times I \to X \) s.t. \( H \circ F = G \).

\( H \) is now a homotopy between \( H|_{S^1 \times 0} = h \) and \( H|_{S^1 \times \{1\}} \) a constant map at \( x_0 \).
The Borsuk-Ulam Theorem for $S^2$: (§57)

If $f : S^2 \to \mathbb{R}^2$ is continuous, then there exists a pair of antipodal points $x, -x \in S^2$ such that $f(x) = f(-x)$.

(similarly for $f : S^n \to \mathbb{R}^n$ in general; case $n=1$ follows from intermediate value theorem, case $n \geq 3$ requires more algebraic topology...).

Next time!