We return to structural properties of top spaces (countability & separation axioms) with a goal of better understanding (next time) which topologies are induced by metrics.

**Countability axioms:** (§3.0)

**Def:** $X$ has a countable basis of neighborhoods at $x \in X$ if $\exists$ countable collection of neighborhoods of $x$, $\{U_n\}_{n \in \mathbb{Z}^+}$, s.t. every neighborhood $V \ni x$ contains at least one of the $U_n$.

A space that has a countable basis of nbhs at each point is said to be **first-countable**

**Ex:** metric spaces are first-countable: at $x \in X$, take $U_n = B_{1/n}(x)$.

(in general, without loss of generality, by intersecting we can assume $U_1 \supset U_2 \supset U_3 \supset \ldots$)

We've mentioned before that this is the property which makes limits of sequences relate accurately to limit points:

Thus: $\ x \in \ X\ \text{top space, } x \in X\ , \text{ if some sequence } x_n \in A \text{ converges to} x\ \text{then } x \in \overline{A}$. If $X$ is first-countable then, conversely, $\ x \in \overline{A}\ \text{then } \exists \text{ sequence } x_n \in A, x_n \to x$.

A stronger requirement is:

**Def:** If the topology on $X$ is generated by a countable basis, then $X$ is said to be **second-countable**.

**Ex:** $\mathbb{R}$ has a countable basis: open intervals $(a,b)$ s.t. $a,b \in \mathbb{Q}$.

$\mathbb{R}^n$ has a countable basis (boxed whose corners have rational coordinate) and even $\mathbb{R}^\omega$ with the product topology has countable basis (same idea: products of intervals which are $(a,b)$, $a,b \in \mathbb{R}$ in finitely many coordinates, $\mathbb{R}$ in all others. This is still countable).

**Ex:** $\mathbb{R}^\omega$ with uniform topology doesn't have countable basis (even though it is first-countable, since topology comes from a metric).

Indeed, $[0,1]^\omega \subset \mathbb{R}^\omega$ uncountable subset, discrete in uniform topology, so every basis of $([\mathbb{R}^\omega, \text{uniform})$ must contain uncountably many basis elements, each containing only one point of $[0,1]^\omega$. 

If $X$ has countable basis then $X$ has a dense subset which is countable.

(Proof: pick one point in each non-empty basis open. The resulting subset $A$ is countable, and intersects every open hence $\overline{A} = X$.)

Ex: $\mathbb{R}$ has a countable dense subset ($\mathbb{Q}$) but doesn't have a countable basis (see HW 2).

Regular and normal spaces (531-32)

Recall: $X$ Hausdorff $\iff$ can separate points: $\forall x,y, \exists U \ni x, \forall y \text{ disjoint open (aka } T_2 \iff T_4; \{x\} \text{ is closed } \forall x \in X)$. 

Stronger separation axioms:

Def: Suppose one-point subsets $\{x\} \subseteq X$ are closed ($T_1$). Then say

- $X$ is regular if $\forall x \in X, V \cap X \text{ closed disjoint from } x, \exists \text{ disjoint open sets } U \ni x, V \ni B$.
- $X$ is normal if $\forall A, B \subseteq X \text{ disjoint closed subsets, } \exists \text{ disjoint open sets } U \supseteq A, V \supseteq B$.

Normal ($T_4$) $\implies$ Regular ($T_3$) $\implies$ Hausdorff ($T_2$) $\implies$ $T_1$

Ex: $\mathbb{R}$ is normal. Indeed: given $A, B \text{ disjoint closed }$:

$\forall a \in A$, $\mathbb{R} - B \text{ open so } \exists \varepsilon_a \ni (a, a + \varepsilon_a) \text{ disjoint from } B$.

$\forall b \in B$, $\exists \varepsilon_b \ni (b, b + \varepsilon_b) \text{ disjoint from } A$.

Take $U = \bigcup \{a, a + \varepsilon_a\} \text{ open } \supseteq A$, $V = \bigcup \{b, b + \varepsilon_b\} \text{ open } \supseteq B$.

Can show $U \cap V = \varnothing$. (Roughly: “because we’ve only extended $A$ & $B$ to the right”)

(usual $\mathbb{R}$ is normal too but need to be a bit more careful when constructing $U \cap V$. See below.)

- $\mathbb{R}^2$ with the product topology is regular but not normal! (hard: see F31 Ex.3)
- $\mathbb{R}$, $\mathbb{R}^n$ are normal. $\mathbb{R}^n$ with product or uniform topology is normal.

If $J$ is uncountable then $\mathbb{R}^J$ with product top is regular but not normal.
Thm: Every regular space w/ countable basis is normal.
(we won’t prove, see Munkres Thm 32.1)

Thm: Every metric space is normal.

Pf: let \( A, B \) disjoint closed \( \subset \langle X, d \rangle \). \( \forall a \in A, \exists \varepsilon_a > 0 \text{ s.t. } B_{\varepsilon_a}(a) \cap X - B. \)
\( \forall b \in B, \exists \varepsilon_b > 0 \text{ s.t. } B_{\varepsilon_b}(b) \cap X - A. \)

Define \( U = \bigcup_{a \in A} B(a, \varepsilon_a / 2) \) and \( V = \bigcup_{b \in B} B(b, \varepsilon_b / 2) \).

why not just \( \varepsilon_a \) & \( \varepsilon_b \)?

\( U \supset A, V \supset B \) are open (unions of open balls).

Claim \( U \cap V = \emptyset \). Indeed: if \( z \in U \cap V \) then \( \exists a \in A, b \in B \) s.t.

\( z \in B(a, \varepsilon_a) \cap B(b, \varepsilon_b) \) hence \( d(a, b) \leq d(a, z) + d(z, b) \)

\[ < \frac{\varepsilon_a + \varepsilon_b}{2} \leq \max(\varepsilon_a, \varepsilon_b). \]

This is a contradiction (eg. if \( d(a, b) < \varepsilon_a \) then \( B(a, \varepsilon_a) \) isn’t disjoint from \( B \) as claimed).

Thm: Every compact Hausdorff space is normal.

Pf: we showed last time: in a Hausdorff space, we can separate points from compact subsets. This implies a compact Hausdorff space is regular.

Recall: \( x \in X, B = X \text{ closed (hence compact) with } x \notin B \):

\( \forall y \in B, \exists U \ni x, V \ni y \text{ disjoint open.} \)

\( B \text{ compact } \Rightarrow \exists y_1, y_2 \in B \text{ s.t. } V = V_{y_1} \cap \ldots \cap V_{y_n} \supset B \)

Then \( U = U_{y_1} \cap \ldots \cap U_{y_n} \) is a neighborhood of \( x \), disjoint from \( V \).

To prove \( X \) is normal, we essentially repeat the same argument.

\( A, B \subset X \text{ disjoint closed (hence compact) subsets } \Rightarrow \)

\( \forall x \in A, \exists U \ni x, V \ni B \text{ disjoint open.} \)

\( A \text{ is compact } \Rightarrow \exists x_1, \ldots, x_n \in A \text{ s.t. } U = U_{x_1} \cap \ldots \cap U_{x_n} \supset A \)

and then \( V = V_{x_1} \cap \ldots \cap V_{x_n} \supset B \text{ open, } U \cap V = \emptyset. \)

We’ve seen that metrizable \( \Rightarrow \) first-countable and normal. There are various metrization theorems which give partial converse.
We'll focus on one that is particularly simple to state (but still requires some clever ideas in order to prove!): the **Urysohn metrization theorem**.

**Theorem:** If $X$ is regular and has a countable basis, then it is metrizable.

The first condition is necessary, but the second one is stronger than necessary. The **Nagata-Smirnov theorem** gives a necessary and sufficient condition:

- $X$ is metrizable iff $X$ is regular and has a "countably locally finite basis".

(we won't discuss this further. Countably locally finite basis means: $B$ is the union of countably many subsets $B_1, B_2, \ldots$, each of which is locally finite i.e. $\forall x \in X, \exists U \ni x$ and intersecting only finitely many elts of $B_i$.)