What is topology? Unlike geometry, which concerns quantitative information about space (distance, volume, ...), topology concerns itself with qualitative properties that are invariant under continuous deformation.

Eg: is it connected? (a single piece)

- simply connected?

\[ \bigcirc \text{ vs. } \bigcirc \]

\[ \text{punctured torus } \sim \quad \bigcirc \]

Eg: how/why is a Möbius band different from a regular band?

\[ \bigcirc \text{ vs. } \bigcirc \]

[orientability? boundary?]

Algebraic topology associates invariants to topological spaces that help tell them apart. We'll get a taste of it in the 2nd part of the course, focusing on the fundamental group.

But first need the language of point-set topology:

- topological spaces, open & closed sets,
- compactness, connectedness.
Topological spaces = sets equipped with data that lets us talk about continuity — notion of "nearness" so we can talk about limits etc.

**Example**: extreme value theorem says: \( f: [a,b] \to \mathbb{R} \) continuous \( \Rightarrow f \) achieves its max and min at some points of \([a,b]\).

This is in fact true for any continuous \( f: X \to \mathbb{R} \) whenever \( X \) is a compact topological space, and is a special instance of

**Theorem**: If \( f: X \to Y \) continuous mapping between topological spaces, & \( X \) compact, then \( f(X) \) is compact.

Since the general notion of topological space is quite abstract, let's start with a more familiar class of examples: **METRIC SPACES**

**Definition**: A metric space \((X,d)\) is a set \(X\) together with a distance function \(d: X \times X \to \mathbb{R}_{\geq 0}\) s.t.

1. For \(p, q \in X\), \(d(p, q) = 0 \iff p = q\)
2. \(d(p, q) = d(q, p)\)
3. For \(p, q, r \in X\), \(d(p, q) \leq d(p, r) + d(r, q)\) (triangle inequality)

**Example**: \(X = \mathbb{R}^n\) with Euclidean distance \(d(x, y) = \left( \sum_{i=1}^{n} (y_i - x_i)^2 \right)^{1/2}\).

If \(Y \subseteq \mathbb{R}^n\) then \((Y, d_{|Y})\) is a metric space ("induced metric").

**Example**: different metrics on \(\mathbb{R}^n\):

\[
\begin{align*}
    d_1(x, y) &= \sum_{i=1}^{n} |y_i - x_i| \\
    d_\infty(x, y) &= \max \{ |y_i - x_i| \}
\end{align*}
\]

Exercises: show \((\mathbb{R}^n, d_1)\) & \((\mathbb{R}^n, d_\infty)\) are metric spaces.

**Open sets**:

**Definition**: \((X, d)\) metric space, \(p \in X\), \(r > 0\) : the open ball of radius \(r\) around \(p\) is \(B_r(p) = \{ q \in X \mid d(p, q) < r \}\) (or neighborhood).

\(U \subseteq X\) is open if \(\forall p \in U\), \(\exists r > 0\) s.t. \(B_r(p) \subseteq U\).

Facts: open balls are open; so are arbitrary unions & finite intersections of open sets. (Homework)
Closed sets & limits:

**Def:** A sequence \( p_1, p_2, \ldots \) in \( X \) converges to a limit \( p \in X \) if \( \forall \epsilon > 0 \ \exists N \in \mathbb{N} \) such that \( \forall n \geq N, \ d(p_n, p) < \epsilon \).

(unique if it exists).

**Def:** A sequence \( p_1, p_2, \ldots \) in \( X \) is Cauchy if \( \forall \epsilon > 0 \ \exists N \in \mathbb{N} \) such that \( \forall m, n \geq N, \ d(p_m, p_n) < \epsilon \).

Exercise: if a sequence converges then it is Cauchy, but not necessarily vice-versa.

A metric space is complete if every Cauchy sequence converges.

Ex: \( \mathbb{R} \) is complete, but \( \mathbb{Q} \) (with induced metric) isn’t complete

**Def:** \( Z \subseteq X \) is closed if its complement \( X \setminus Z \) is open.

Most subsets of \( X \) are neither open nor closed!!

\( \emptyset \) and \( X \) are both open and closed!

**Prop.** \( Z \subseteq X \) is closed if and only if:

\( \forall \) sequence \( \{p_n\} \) in \( Z \) which converges to a limit \( p \in X \), then \( p \in Z \).

\( \triangle \Rightarrow \) true in all topological spaces, but \( \Leftarrow \) only holds in sufficiently nice ones (such as metric spaces)

**Proof:** if \( Z \) is not closed then \( X \setminus Z \) not open, i.e. \( \exists p \in X \setminus Z \) st. \( \forall r > 0, \ B_r(p) \nsubseteq X \setminus Z \)

For this point \( p \), \( \forall n \geq 1, \ \exists p_n \in Z \) with \( d(p_n, p) < \frac{1}{n} \).

This gives a sequence \( p_n \to p \), \( p_n \in Z \), \( p \notin Z \).

\( \cdot \) Conversely, if \( \exists p_n \in Z \), \( p \in X \setminus Z \), \( p_n \to p \), then

\( \forall r > 0 \ \exists N \in \mathbb{N} \) such that \( \forall n \geq N, \ d(p_n, p) < r \).

So \( N_r(p) \) contains points of \( Z \), hence \( N_r(p) \nsubseteq X \setminus Z \).

Hence \( X \setminus Z \) isn’t open, i.e. \( Z \) isn’t closed.

**Continuity:** **Def:** \((X, d_X), (Y, d_Y)\) metric spaces. \( f : X \to Y \) is continuous if

\[ \forall p \in X, \ \forall \epsilon > 0, \ \exists \delta > 0 \text{ st. } d_X(p, x) < \delta \Rightarrow d_Y(f(p), f(x)) < \epsilon. \]

\( \exists \) 8-ball \( p \) \( f \) \( \Rightarrow \) 8-ball given \( \epsilon \)-ball.
Theorem: \( f: X \to Y \) is continuous iff \( \forall U \subset Y \text{ open}, \ f^{-1}(U) \subset X \text{ is open.} \)

**Proof.** Assume \( f \) continuous, let \( U \subset Y \) open, let \( p \in f^{-1}(U) \), i.e. \( f(p) \in U \).

Want: \( \exists \delta > 0 \text{ st. } B_\delta(p) \subset f^{-1}(U). \)

Know: \( \exists \epsilon > 0 \text{ st. } B_\epsilon(f(p)) \subset U. \) (since \( U \) open).

By continuity, \( \exists \delta > 0 \text{ st. } d(p,x) < \delta \Rightarrow f(x) \in B_\epsilon(f(p)) \subset U. \)

Hence \( B_\delta(p) \subset f^{-1}(U). \) So \( f^{-1}(U) \) is open.

Conversely, assume \( U \subset Y \) open \( \Rightarrow f^{-1}(U) \) open.

Fix \( p \in X, \epsilon > 0. \) \( B_\epsilon(f(p)) \) is open in \( Y \), so \( f^{-1}(B_\epsilon(f(p))) \exists \delta > 0 \text{ st. } B_\delta(p) \subset f^{-1}(B_\epsilon(f(p))). \)

This means \( d(p,x) < \delta \Rightarrow x \in B_{\epsilon}(f(p)) \Rightarrow f(x) \in B_{\epsilon}(f(p)) \).

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* Our goal will be to reformulate / generalize all this in the context of **topological spaces**, i.e. sets equipped with a topology which may or may not come from a metric.

**Def.** A topology \( T \) on a set \( X \) is collection of subsets of \( X \), which we'll declare to be the open sets in \( X \). Needs to satisfy axioms:

- \( \emptyset \in T, X \in T \)
- any union of elements of \( T \) is in \( T \)
- the intersection of finitely many elements of \( T \) is in \( T \).

Why bother? One answer: many natural topologies do not come from a metric!

E.g., in analysis:

- *on space of (bounded) functions \( f: X \to \mathbb{R}, \)
  - uniform convergence topology *comes* from a metric \( (d(f,g) = \sup_x |f(x) - g(x)|) \)
  - but pointwise convergence \( f_n \to f \iff \forall x \in X, f_n(x) \to f(x) \) doesn't. ("product topology")
- \( C^\infty \) topology on smooth functions \( \mathbb{R} \to \mathbb{R} \) doesn't come from a metric either.