Math 112 Homework 8 Solutions

Problem 1.
To show \( f \not\in \mathcal{R} \) for any \([a, b]\), we show that for all partitions \( P \), \( U(P, f) = b - a \) but \( L(P, f) = 0 \), which implies that the lower and upper integrals or \( f \) are not equal and hence \( f \not\in \mathcal{R} \).

Fix a partition \( P = \{a = x_0 < x_1 < \cdots < x_n = b\} \). Note: the definition on p. 120 allows \( x_0 \leq x_1 \leq \cdots \leq x_n \), but if \( x_i = x_{i+1} \) then \( \Delta x_i = 0 \), so we can discard any point that appears more than once). Each interval \([x_{i-1}, x_i]\) contains a rational number (by the density of the rational numbers, Theorem 1.20(b)), so \( M_i = 1 \) for all \( i \). On the other hand, each interval \([x_{i-1}, x_i]\) also contains irrational numbers, so \( m_i = 0 \) for all \( i \). This implies that \( L(P, f) = \sum_{i=1}^{n} m_i \Delta x_i = 0 \) and \( U(P, f) = \sum_{i=1}^{n} M_i \Delta x_i = \sum_{i=1}^{n} (x_i - x_{i-1}) = b - a \), which implies that \( f \) is not Riemann-integrable.

Problem 2.
Consider the partition \( P_n = \{1, 2, \ldots, n\} \) of the interval \([1, n]\). Since \( f \) is monotonically decreasing, \( \inf_{[i-1,i]} f = f(i) \), while \( \Delta x_i = 1 \) for all \( i \), so we have \( L(P_n, f) = \sum_{i=1}^{n} f(i) \) and \( U(P_n, f) = \sum_{i=1}^{n} f(i - 1) = \sum_{i=1}^{n-1} f(i) \). Since \( L(P_n, f) \leq \int_{1}^{n} f \ dx \leq U(P_n, f) \), we have the inequalities \( \sum_{i=2}^{n} f(i) \leq \int_{1}^{n} f \ dx \leq \sum_{i=1}^{n-1} f(i) \). By comparison, we get that the integral converges if and only if the series converges.

More precisely, assume that \( \sum_{i=1}^{\infty} f(i) \) converges; then the integrals \( I_n = \int_{1}^{n} f \ dx \leq \sum_{i=1}^{n-1} f(i) \leq \sum_{i=1}^{\infty} f(i) \) are bounded, and so \( \int_{1}^{\infty} f \ dx \) is convergent (remark that \( \{I_n\} \) is a bounded monotonically increasing sequence, and that any integral of the form \( \int_{1}^{A} f \ dx \) can be bounded between \( I_n \) and \( I_{n+1} \) for some value of \( n \)). Conversely, assume that \( \int_{1}^{\infty} f \ dx \) converges; then the sums \( \sum_{i=2}^{n} f(i) \leq \int_{1}^{n} f \ dx \leq \int_{1}^{\infty} f \ dx \) are bounded, so the series \( \sum_{i=2}^{n} f(i) \) is convergent (it is a series of non-negative terms and its partial sums are bounded); adding the single term \( f(1) \) to the series, \( \sum_{i=1}^{\infty} f(i) \) is also convergent.

Problem 3.
(a) First observe that, since \( \frac{1}{p} + \frac{1}{q} = 1 \) and \( p, q \) are positive, we must have \( \frac{1}{p}, \frac{1}{q} < 1 \), i.e. \( p, q > 1 \). For a fixed value of \( v \geq 0 \), let \( \phi : [0, \infty) \to \mathbb{R} \) be the function defined by \( \phi(u) = \frac{1}{p} u^p - uv \). The function \( \phi \) is differentiable and \( \phi'(u) = u^{p-1} - v \). Let \( \alpha = v^{1/(p-1)} > 0 \), \( \phi' \) is a strictly increasing function, so \( \phi' \) takes negative values over \([0, \alpha]\) and positive values over \((\alpha, +\infty)\). Hence, by the mean value theorem, \( \phi \) is strictly decreasing over the interval \([0, \alpha]\) and strictly increasing over the interval \([\alpha, +\infty)\). In particular, \( \forall u \geq 0 \) we have \( \phi(u) \geq \phi(\alpha) \), with equality if and only if \( u = \alpha \). Observe that \( v^{-1} = 1 - \frac{1}{p} = \frac{1}{p} \); therefore \( \phi(\alpha) = \frac{1}{p} \alpha^p - p\alpha = \frac{1}{p} v^{p/(p-1)} - v^{1/(p-1)+1} = \frac{1}{p} v^{p/(p-1)} - \frac{1}{q} v^q \). We conclude that, for every \( u, v \geq 0 \), \( \frac{1}{p} u^p - uv \geq \frac{1}{q} v^q \), or equivalently, \( uv \leq \frac{1}{p} u^p + \frac{1}{q} v^q \), with equality if and only if \( u = v^{1/(p-1)} \), i.e. \( u^p = v^{p/(p-1)} = v^q \).

(b) Note that, if \( f, g \in \mathcal{R} \), then by Theorems 6.11 and 6.13, \( f^p, g^q \) and \( f g \) are also integrable. By part (a), we have the inequality \( f(x) g(x) \leq \frac{1}{p} f(x)^p + \frac{1}{q} f(x)^q \) for every \( x \in [a, b] \). Therefore, by Theorem 6.12, we have \( \int_{a}^{b} f g \ dx \leq \int_{a}^{b} \left( \frac{1}{p} f^p + \frac{1}{q} f^q \right) dx = \frac{1}{p} \int_{a}^{b} f^p \ dx + \frac{1}{q} \int_{a}^{b} g^q \ dx = \frac{1}{p} + \frac{1}{q} = 1 \), which is the desired result.

(c) Let \( I = (\int_{a}^{b} f^p \ dx)^{1/p} \) and \( J = (\int_{a}^{b} |g|^q \ dx)^{1/q} \). Observe that \( \int_{a}^{b} \left( \frac{1}{p} |f|^p \right) \ dx = \frac{1}{p} \int_{a}^{b} |f|^p \ dx = 1 \), and similarly \( \int_{a}^{b} \left( \frac{1}{q} |g|^q \right) \ dx = \frac{1}{q} \int_{a}^{b} |g|^q \ dx = 1 \). Therefore, applying the result of (b) to the functions
\[ \frac{1}{2} |f| + \frac{1}{2} |g| \], we get \[ \int_a^b \frac{1}{2} |fg| \, dx \leq 1 \], or equivalently \[ \int_a^b |fg| \, dx \leq IJ \). Using Theorem 6.13, which remains true for complex-valued functions (cf. Theorem 6.25), we have \[ | \int_a^b fg \, dx | \leq \int_a^b |fg| \, dx \leq IJ \], which completes the proof.

**Problem 4.**

(a) For \( n = 0 \), the formula becomes \( f(x) = f(a) + \int_a^x f'(t) \, dt \), which is the fundamental theorem of calculus (Theorem 6.21).

(b) By comparing the given expressions for \( n - 1 \) and \( n \), one sees that it is sufficient to prove that \( R_n(x) = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f^{(n)}(t) \, dt = \frac{1}{n!} \int_a^x F(t)G'(t) \, dt \)

\[ \frac{F(x)G(x) - F(a)G(a)}{(n-1)!} - \frac{1}{(n-1)!} \int_a^x F'(t)G(t) \, dt \]

\[ 0 + \frac{f^{(n)}(a)(x-a)^n}{n(n-1)!} + \frac{1}{n(n-1)!} \int_a^x f^{(n+1)}(t)(x-t)^n \, dt = \frac{f^{(n)}(a)}{n!} (x-a)^n + R_{n+1}(x). \]

The result follows by induction on \( n \).

**Problem 5.**

Assume that \( \{f_n\} \) converges uniformly to \( f \) and \( \{g_n\} \) converges uniformly to \( g \). First observe that \( f \) is bounded. Indeed, there exists \( N \) such that if \( n \geq N \) then \( |f_n(x) - f(x)| \leq 1 \) \( \forall x \in E \). Therefore, setting \( M = 1 + \sup_{x \in E} |f_N(x)| \), for every \( x \in E \) we have \( |f(x)| \leq |f_N(x)| + 1 \leq M \). Similarly, \( g \) is bounded by a constant \( M' \).

Fix \( \epsilon > 0 \); decreasing \( \epsilon \) if necessary we can assume that \( \frac{\epsilon}{2M} < 1 \). Since \( f_n \to f \) uniformly, there exists \( N_1 \) such that \( \forall n \geq N_1, \forall x \in E, |f_n(x) - f(x)| \leq \frac{\epsilon}{2M} \). Similarly there exists \( N_2 \) such that \( \forall n \geq N_2, \forall x \in E, |g_n(x) - g(x)| \leq \frac{\epsilon}{2M} \). Let \( n \geq N = \max(N_1,N_2) \) and \( x \in E \): we have

\[ |f_n(x)g_n(x) - f(x)g(x)| = |f_n(x)(g_n(x) - g(x)) + (f_n(x) - f(x))g(x)| \leq |f_n(x)||g_n(x) - g(x)| + |f_n(x) - f(x)||g(x)| \leq (|f(x)| + \frac{\epsilon}{2M}) \frac{\epsilon}{2M} + \frac{\epsilon}{2M} |g(x)| \leq (M + 1) \frac{\epsilon}{2M} + M \frac{\epsilon}{2M} = \epsilon. \]

So we have proved that \( f_ng_n \) converges uniformly to \( fg \).

The assumption that \( f_n \) and \( g_n \) are bounded is necessary: consider e.g. \( E = (0, +\infty) \), \( f_n(x) = x \), and \( g_n(x) = \frac{1}{x} + \frac{1}{n} \). It is clear that \( f_n \) converges uniformly to \( f(x) = x \) and \( g_n \) converges uniformly to \( g(x) = \frac{x}{x} \). We have \( f_n(x)g_n(x) = 1 + \frac{x}{n} \), which converges to 1 as \( n \to \infty \) for every value of \( x \). However, \( f_ng_n \) does not converge uniformly to 1 over \((0, +\infty), in fact \( f_ng_n - 1 \) is not even bounded (consider \( x \to 10)\).
Problem 6.

If $x > 0$ then $\left| \frac{1}{1+n^2x} \right| \leq \frac{1}{n^2x} = \frac{x^{-1}}{n^2}$. Since $\sum \frac{1}{n^2}$ converges, by the comparison criterion $\sum \frac{1}{1+n^2x}$ converges absolutely for all $x > 0$. If $x = 0$ the series is divergent since $\frac{1}{1+n^2x} = 1$.

Given any constant $a > 0$, we have the inequality $\left| \frac{1}{1+n^2x} \right| \leq \frac{1}{n^2x} \leq \frac{a^{-1}}{n^2}$ for all $x \in [a, +\infty)$. Observe that the series $\sum \frac{a^{-1}}{n^2}$ is convergent; therefore, by Theorem 7.10, the series $\sum \frac{1}{1+n^2x}$ converges uniformly on the interval $[a, +\infty)$ for all $a > 0$.

We now consider an interval of the form $(0, a)$: fix an integer $N$, and let $x \in (0, a)$ be such that $x < \frac{1}{N^2}$. Then $\frac{1}{1+N^2x} > \frac{1}{1+N^2} = \frac{1}{2}$. This proves that $\frac{1}{1+n^2x}$ does not converge uniformly to 0 over $(0, a)$, which is enough to derive a contradiction to the Cauchy criterion (Theorem 7.8): therefore the series does not converge uniformly on any of the intervals $(0, a)$ for $a > 0$.

Alternatively: letting $f_n(x) = \sum_{k=1}^{n} \frac{1}{1+k^2x}$, we set $A_n = \lim_{t \to 0} f_n(t) = n$. If one assumes the sequence $\{f_n\}$ to converge uniformly over $(0, a)$, by Theorem 7.11 the sequence $\{A_n\}$ must be convergent (and its limit equals $\lim_{t \to 0} f(t)$). This gives a contradiction, so the convergence is not uniform over $(0, a)$.

Since the individual terms in the series are continuous functions and since the convergence is uniform over $[a, +\infty)$, by Theorem 7.12 the function $f$ is continuous over $[a, +\infty)$ for all $a > 0$. Therefore $f$ is continuous at every point of $(0, +\infty)$.

The function $f$ is not bounded: given any integer $N$, if $x \in (0, \frac{1}{N^2})$ then for every $n \leq N$ we have

$$
\frac{1}{1+n^2x} > \frac{1}{1+n/\sqrt{n}} \geq \frac{1}{2},
$$

so $\sum_{n=1}^{N} \frac{1}{1+n^2x} > \frac{N}{2}$, and therefore $f(x) > \frac{N}{2}$. 
