Math 112 Homework 5 Solutions

Problem 1.
Since \((\sqrt{a_n} - \frac{1}{n})^2 = a_n + \frac{1}{n^2} - \frac{2}{n} \sqrt{a_n}\), we have

\[
0 \leq \frac{1}{n} \sqrt{a_n} \leq \frac{1}{2}(a_n + \frac{1}{n^2}).
\]

The series \(\sum a_n\) converges by assumption, and the series \(\sum \frac{1}{n^2}\) converges, so by the comparison criterion the series \(\sum \frac{1}{n} \sqrt{a_n}\) converges.

Problem 2.

a) \(\limsup_{n \to \infty} \sqrt[n]{n^3} = \limsup_{n \to \infty} n^{3/n} = \limsup_{n \to \infty} (n^{1/n})^3 = 1\), so \(R = 1\).

b) Note that \(\sum_{n=0}^{\infty} \frac{2^n}{n!} z^n = \sum_{n=0}^{\infty} \left(\frac{2z}{n!}\right)^n = e^{2z}\). We know that \(e^z\) has \(R = \infty\), i.e. \(\limsup(n!)^{-1/n} = 0\) (see Example 3.40(b)). Therefore \(\limsup(2^n/n!)^{1/n} = 2 \limsup(1/n!)^{1/n} = 0\), and \(R = \infty\) again.

c) Observe that, for any \(k \in \mathbb{Z}\), \(\lim_{n \to \infty} (n^k/n!) = 1\). This is because \(\lim n^{1/n} = 1\) (Theorem 3.20(c)), so \(\lim_{n \to \infty} n^{k/n} = 1\). Hence, we have \(\limsup_{n \to \infty} (2^n/n^2)^{1/n} = 2 \limsup_{n \to \infty} n^{-2/n} = 2\), so \(R = 1/2\).

d) Similarly, \(\limsup_{n \to \infty} (n^3/3^n)^{1/n} = 1/3\), so \(R = 3\).

Problem 3.

a) Observe that, if \(a_n \geq 1\), then \(\frac{a_n}{1 + a_n} \geq \frac{a_n}{a_n + a_n} = \frac{1}{2}\), while if \(a_n \leq 1\), then \(\frac{a_n}{1 + a_n} \geq \frac{a_n}{1 + 1} = \frac{a_n}{2}\). We consider the following alternative.

Case 1: there exist infinitely many values of \(n\) such that \(a_n > 1\). Then the corresponding values of \(\frac{a_n}{1 + a_n}\) are all greater than \(\frac{1}{2}\), so \(\frac{a_n}{1 + a_n} \not\to 0\), and therefore \(\sum \frac{a_n}{1 + a_n}\) diverges.

Case 2: for all but finitely many values of \(n\), we have \(a_n \leq 1\) and therefore \(\frac{a_n}{1 + a_n} \geq \frac{a_n}{2}\). Since a finite number of terms cannot affect the behavior of the series, by the comparison criterion we can conclude that, since \(\sum \frac{a_n}{2}\) is divergent, the series \(\sum \frac{a_n}{1 + a_n}\) is also divergent.

(Shorter alternate solution: if \(a_n \not\to 0\) then \(\frac{a_n}{1 + a_n} \not\to 0\) and then \(\sum \frac{a_n}{1 + a_n}\) diverges. If \(a_n \to 0\) then there exists \(N\) such that, for \(n \geq N\), \(a_n \leq 1\) and therefore \(\frac{a_n}{1 + a_n} \geq \frac{a_n}{2}\) \(\forall n \geq N\), which gives divergence by the comparison criterion).

b) Since \(a_n > 0\), the sequence \(\{s_n\}\) is monotonically increasing, and therefore \(\frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} = \frac{a_{N+1} + \cdots + a_{N+k}}{s_{N+k}} = \frac{s_{N+k} - s_N}{s_{N+k}} = 1 - \frac{s_N}{s_{N+k}}\).

Observe that the divergence of the series \(\sum a_n\) implies that the monotonically increasing sequence \(s_n\) is not bounded (by Theorem 3.24). In particular, given any integer \(N\), the constant \(2s_N\) is not an upper bound for \(\{s_n\}\), i.e. there exists \(m\) such that \(s_m \geq 2s_N\); obviously \(m > N\), so we can write \(m = N + k\) for some positive integer \(k\). Therefore, for every integer \(N\), there exists an integer \(k\) such that \(s_{N+k} \geq 2s_N\), i.e. \(1 - \frac{s_N}{s_{N+k}} \geq \frac{1}{2}\). As a consequence, we obtain: \(\forall N \exists k\) such that \(\sum_{n=N+1}^{N+k} \frac{a_n}{s_n} \geq \frac{1}{2}\).

Assume that the series \(\sum \frac{a_n}{s_n}\) is convergent: then by the Cauchy criterion (Theorem 3.22) there exists \(N\) such that, \(\forall m \geq n \geq N\), \(|\sum_{k=n}^{m} \frac{a_k}{s_k}| \leq \frac{1}{3}\). This contradicts the previously obtained statement; therefore \(\sum \frac{a_n}{s_n}\) diverges.

1
c) Since \( s_n = s_{n-1} + a_n \geq s_{n-1} \), we have \( \frac{1}{s_{n-1}} - \frac{1}{s_n} = \frac{s_n - s_{n-1}}{s_{n-1} s_n} = \frac{a_n}{s_{n-1} s_n} \geq \frac{a_n}{s_n^2} \). Therefore
\[
\sum_{n=1}^{N} \frac{a_n}{s_n^2} \leq \frac{a_1}{s_1^2} + \sum_{n=2}^{N} \left( \frac{1}{s_{n-1}} - \frac{1}{s_n} \right) = \frac{1}{a_1} + \left( \frac{1}{s_1} - \frac{1}{s_2} \right) + \cdots + \left( \frac{1}{s_{N-1}} - \frac{1}{s_N} \right) = \frac{2}{a_1} - \frac{1}{s_N} < \frac{2}{a_1}.
\]
So the partial sums of the series of non-negative terms \( \sum \frac{a_n}{s_n^2} \) are bounded by the constant \( \frac{2}{a_1} \), and therefore by Theorem 3.24 the series \( \sum \frac{a_n}{s_n^2} \) is convergent.

d) The series \( \sum \frac{a_n}{1 + n a_n} \) can be either divergent or convergent. Divergent examples are easy to come by (e.g., if \( a_n = 1 \) then \( \frac{a_n}{1 + n a_n} = \frac{1}{n+1} \) and \( \sum \frac{1}{n+1} \) is divergent; if \( a_n = \frac{1}{n} \) then \( \frac{a_n}{1 + na_n} = \frac{1}{2n} \) and \( \sum \frac{1}{2n} \) is divergent).

An example where \( \sum a_n \) diverges but \( \sum \frac{a_n}{1 + n a_n} \) converges is given by setting \( a_n = 1 \) if there exists \( k \) such that \( n = k^2 \), and \( a_n = \frac{1}{n^2} \) otherwise. Because infinitely many terms are equal to 1 the sequence \( \{a_n\} \) does not converge to 0; therefore \( \sum a_n \) is divergent. Meanwhile, if \( n = k^2 \) then \( \frac{a_n}{1 + n a_n} = \frac{1}{k^2 + 1} < \frac{1}{k^2} \), while in the other case \( \frac{a_n}{1 + n a_n} < a_n = \frac{1}{n^2} \). Therefore the partial sums of the series \( \sum \frac{a_n}{1 + n a_n} \) are bounded: \( \sum_{n=1}^{N} \frac{a_n}{1 + n a_n} \leq \sum_{k^2 \leq N} \frac{1}{k^2} + \sum_{n=1}^{N} \frac{1}{n^2} \leq \sum_{k=1}^{\infty} \frac{1}{k^2} + \sum_{n=1}^{\infty} \frac{1}{n^2} \) (recall \( \sum \frac{1}{n^2} \) is convergent). By Theorem 3.24 we conclude that in this example \( \sum \frac{a_n}{1 + n a_n} \) is convergent.

The case of the series \( \sum \frac{a_n}{1 + n^2 a_n} \) is easier: it is always convergent, by the comparison criterion, because \( 0 \leq \frac{a_n}{1 + n^2 a_n} \leq \frac{1}{n^2} \).