Serre's conjecture on the Artin character for a regular local ring

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Abstract

We prove Serre's conjecture on the Artin character of a finite group G acting on a regular local ring $\mathcal{A}$ with isolated fixed points. The case of positive characteristic is due to Abe [2]. In mixed characteristic we use an equivariant deformation to reduce to the case where $\mathcal{A}$ has an isolated singularity, in which case we can make use of the results of Beraldo and Pippi on the twisted conductor formula.

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1 Introduction

Let $\mathcal{A}$ be a regular local ring with maximal ideal $\mathfrak{m}$ and residue field $k = \mathcal{A}/\mathfrak{m}$, and let G be a finite group acting on $\mathcal{A}$. Write $\mathcal{X} = \operatorname{Spec}\mathcal{A}$. Motivated by the form of the local terms in the *global* Lefschetz trace formula, Serre conjectured that the multiplicities of the fixed point for $g \neq 1$ could be used to define a character of G, analogous to the classical Artin character for local fields. More precisely, for $g \in G$, put

$$I_g(\mathcal{A}) = (g(a) - a \mid a \in \mathcal{A}).$$

We write I_g when the ring is understood. We make the following assumptions:

(i) the inclusion $\mathcal{A}^G \subset \mathcal{A}$ induces an isomorphism $\mathcal{A}^G/(\mathcal{A}^G \cap \mathfrak{m}) \simeq k$;

(ii) $\mathcal{A}/I_g(\mathcal{A})$ has finite length for $g \neq 1$.

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For $g \neq 1$, write $i_{\mathcal{A}}(g) = \operatorname{length}_{\mathcal{A}}(\mathcal{A}/I_g(\mathcal{A}))$. Define

$$a_{\mathcal{A},G}(g) = \begin{cases} -i_{\mathcal{A}}(g), & g \neq 1, \\ \sum_{h \neq 1} i_{\mathcal{A}}(h), & g = 1. \end{cases} \tag{1.1}$$

This is a central function on G. For a discrete valuation ring it is the classical Artin character. The following assertion was conjectured by Serre in 1960 [19, §6].

Theorem 1.1. *For every prime ℓ invertible in $\mathcal{A}$, $a_{\mathcal{A},G}$ is the character of a finite-dimensional $\mathbf{Q}_\ell$-representation of G.*

The case of a ring containing a field of positive characteristic is due to Abe [2]. In residue characteristic zero the action is tame, and the assertion follows from an easy calculation. There is an approach to the conjecture via vanishing cycles which goes back to Kato, Shuji Saito and Takeshi Saito [13, §5], as explained by Abe [2, Introduction and §2.6]. In particular, for arithmetic surfaces, a proof along these lines appears in Abbes [1, Lemma 1.2 and Corollary 1.3].

We treat the mixed-characteristic case by reducing to the twisted conductor formula of Beraldo–Pippi [5, §8.5]. After completion and approximation, the action on $\mathcal{X}$ comes from a regular finite type G-scheme $X = \operatorname{Spec} A$ over a mixed characteristic discrete valuation ring C, such that for $g \in G$, $g \neq 1$, the fixed scheme $X^g \subset X$ is supported at a closed point $x \in X$. We use a deformation to replace X by a regular flat C-scheme X' whose nonsmooth locus is supported at x. This allows us to apply the results of Beraldo and Pippi [5].

Let us explain this in more detail, beginning with the deformation argument. We show that X, as above, may be chosen finite free over a smooth C-algebra R, on which G acts trivially, and such that $X \to \operatorname{Spec} R$ is totally ramified over the image b of x. We must construct X' without changing the algebras $\mathcal{O}_{X,x}/I_g$. Consider the R-scheme Z parametrizing algebra structures with G-action on a free module of rank $\operatorname{rank}_R A$, and let $W \to Z$ be its universal family. The algebra A, with a choice of basis, defines a section $z : \operatorname{Spec} R \to Z$. The fibers of $X \to \operatorname{Spec} R$ are complete intersections, since this morphism is finite flat between regular schemes. Away from b they also have free G-action. It follows that Z/R is smooth along $z(\operatorname{Spec} R \setminus \{b\})$, and W/R is smooth above this locus (Lemma 4.1).

The parameter lemma of §3 gives a smooth affine R-scheme T, a section $e : \operatorname{Spec} R \to T$, and a morphism $f : T \to Z$ with $f \circ e = z$, such that f is smooth along $e(\operatorname{Spec} R \setminus \{b\})$. Put $Y = W \times_Z T$. The projection $Y \to W$ is then smooth along $X \setminus \{x\}$, so Y/C is smooth there. Let r be the relative dimension of T/R at $e(b)$. Near $e(b)$, the section is cut out by relative parameters $t_1, \dots, t_r$. Their pullbacks to Y form a regular sequence with quotient $\mathcal{O}_{X,x}$, so Y is regular at x. After shrinking, we obtain a regular flat affine C-scheme $Y = \operatorname{Spec} D$ with

$$X = V(t_1, \dots, t_r), \qquad t_i \in D^G.$$

Let $H \subset D$ define the reduced nonsmooth locus of Y/C, and let $\mathfrak{m}_x \subset D$ be the ideal of x. For $g \neq 1$, put $J_g = (g(a) - a \mid a \in D)$, and set $J = \bigcap_{g \neq 1} J_g$. We replace the equations t_i by

$$t_i' = t_i + \delta_i, \qquad \delta_i \in (HJ\mathfrak{m}_x^2)^G, \qquad X' = V(t_1', \dots, t_r').$$

As $t_i' = t_i$ modulo $\mathfrak{m}_x^2$, X' is regular at x. The factor J ensures that, for every $g \neq 1$,

$$D/(t_1', \dots, t_r', J_g) = D/(t_1, \dots, t_r, J_g).$$

Thus the fixed schemes agree, and their multiplicities $i_{\mathcal{A}}(g)$ are unchanged. The factor H similarly gives $X' \cap V(H) = X \cap V(H)$, so X' meets the nonsmooth locus of the ambient scheme only at x. A Bertini argument shows that the new equations $(t'_1, \dots, t'_r)$ may be chosen transverse on $U = Y \setminus \big(V(J) \cup V(H) \cup \{x\}\big)$, so that X'/C is smooth outside x.

The nonsmooth locus of X'/C, with its reduced structure, is proper over C, since it is supported at the closed point x. This is the properness hypothesis needed for the *twisted Bloch conductor formula* of Beraldo and Pippi [5, §8.5.6]. For $g \neq 1$, we apply their formula to the structure sheaves of the diagonal Δ and the graph Γ_g of g in $X' \times_C X'$. It compares a localized intersection number with two cohomological traces arising from vanishing cycles.[1] Lemma 5.1 identifies the intersection number with

$$\operatorname{length}_{\mathcal{O}_{X',x}}(\mathcal{O}_{X',x}/I_g) = i_{\mathcal{A}}(g).$$

Now put $K = \operatorname{Frac}(C)$ and choose a sufficiently large finite Galois extension L/K, with group $H_0 = \operatorname{Gal}(L/K)$. The two cohomological terms yield finite virtual representations V_0 of G and V_1 of $G \times H_0$ over $\mathbf{Q}_\ell$. To identify their traces, we use the duality maps in Beraldo and Pippi's construction: replacing the diagonal by the graph inserts the geometric action of g in the trace (Lemma 5.2). Let Art_{H_0} be the classical Artin representation associated with L/K. It is defined over $\mathbf{Q}_\ell$ by Serre's theorem [20, §19.2, Theorem 44(i)]. The conductor formula becomes

$$i_{\mathcal{A}}(g) = \operatorname{Tr}(g, V_0) - \operatorname{Tr}\big(g, \operatorname{Hom}_{H_0}(\operatorname{Art}_{H_0}, V_1)\big) \qquad (g \neq 1).$$

Consequently

$$V := -V_0 + \operatorname{Hom}_{H_0}(\operatorname{Art}_{H_0}, V_1)$$

is an integral virtual representation over $\mathbf{Q}_\ell$, and satisfies $\operatorname{Tr}(g, V) = -i_{\mathcal{A}}(g)$ for every $g \neq 1$. A straightforward argument which goes back to Serre's original treatment then shows that

$$W = V - \dim(V^G)\mathbf{Q}_\ell[G]$$

is a genuine (not just virtual) representation, and has character $a_{\mathcal{A},G}$.

Although we have only considered the case of mixed characteristic, a simplified form of the above argument does lead to a proof of Serre's conjecture in characteristic p. Let us compare this argument with Abe's proof. In this case, it was observed by Kato–Saito–Saito [13, Lemma 5.3] that one can reduce to $X = \operatorname{Spec} A$ of finite type over an algebraically closed field k, admitting a G-invariant map $f : X \to \mathbb{A}^1_k$ smooth away from x. The argument of §5, using the results of Beraldo-Pippi, then applies directly over $C = \widehat{\mathcal{O}}_{\mathbb{A}^1_k, f(x)}$, without the using the deformation results of §§3–4. This gives an alternative to Abe's proof, which instead proves a trace formula on vanishing cycles by combining the results of Lu–Zheng [17, §§2.4–2.5], with an argument using Laumon's local Fourier transform [2, Proposition 2.5(2)]. The main extra difficulty in mixed characteristic is that one has much less choice for the map from $\operatorname{Spec} A$ to a trait.

AI use

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[1] Compare the argument of [16, Theorem 3.3], which relates nearby-cycle traces to fixed points of local lifts on the generic fiber. Both arguments use passage to the generic fiber to study possibly degenerate fixed points on the special fiber.

2 Reduction to finite type algebras

For a separated scheme X with G-action and $g \neq 1$, let $I_g(X) \subset \mathcal{O}_X$ be the ideal sheaf of the fixed scheme $\mathrm{Fix}_X(g)$. If $X = \operatorname{Spec} A$, we also write $I_g(A)$ for the corresponding ideal of A, generated by the elements $g(a) - a$ for $a \in A$.

We first record a lemma asserting that the fixed point scheme can be detected by a finite length quotient of $\mathcal{A}$.

Lemma 2.1. *Let $(\mathcal{A}, \mathfrak{m})$ and $(\mathcal{A}', \mathfrak{m}')$ be Noetherian local rings with G-action. Fix $g \in G$ and a positive integer N. Suppose there is a G-equivariant isomorphism $\alpha : \mathcal{A}/\mathfrak{m}^N \simeq \mathcal{A}'/(\mathfrak{m}')^N$. If $L = \operatorname{length}_{\mathcal{A}}(\mathcal{A}/I_g(\mathcal{A})) < N$, then this isomorphism induces*

$$\mathcal{A}/I_g(\mathcal{A}) \simeq \mathcal{A}'/I_g(\mathcal{A}').$$

Proof. If $L = 0$, equivariance of α gives $I_g(\mathcal{A}') + (\mathfrak{m}')^N = \mathcal{A}'$, whence $I_g(\mathcal{A}') = \mathcal{A}'$. Both quotients are then zero. Assume $L > 0$. One has $\mathfrak{m}^L \subset I_g(\mathcal{A})$, so the existence of α implies $(\mathfrak{m}')^L \subset I_g(\mathcal{A}') + (\mathfrak{m}')^N$. In $\mathcal{B}' = \mathcal{A}'/I_g(\mathcal{A}')$, with maximal ideal $\mathfrak{n}'$, this gives $(\mathfrak{n}')^L = (\mathfrak{n}')^N$. Since $N > L$, Nakayama's lemma, applied to $(\mathfrak{n}')^L$, gives $(\mathfrak{n}')^L = 0$. Hence α induces the claimed isomorphism. □

Under the hypotheses of Theorem 1.1, completion preserves regularity and the residue field, and the G-action extends to $\widehat{\mathcal{A}}$. Lemma 2.1 shows that completion preserves $\mathcal{A}/I_g(\mathcal{A})$ for $g \neq 1$. The residue-field condition also persists, since $\mathcal{A}^G \subset (\widehat{\mathcal{A}})^G$ already surjects onto k. We may therefore replace $\mathcal{A}$ by its completion.

Suppose now that $\mathcal{A}$ is complete, and put $\mathcal{B} = \mathcal{A}^G$. We claim that $\mathcal{A}$ is finite over $\mathcal{B}$. Choose generators $x_1, \dots, x_r$ of $\mathfrak{m}$, and let $\mathfrak{q} \subset \mathcal{B}$ be generated by the nonleading coefficients of the orbit polynomials $\prod_{g \in G}(T - g(x_i))$. Then $x_i^{|G|} \in \mathfrak{q}\mathcal{A}$ for every i, so $\mathfrak{q}\mathcal{A}$ is $\mathfrak{m}$-primary. The equality of residue fields implies that $\mathcal{A}/\mathfrak{q}\mathcal{A}$ is a finite $\mathcal{B}$-module. Choose lifts $v_1, \dots, v_s \in \mathcal{A}$ of generators. Successive substitution in $\mathcal{A} = \sum_j \mathcal{B} v_j + \mathfrak{q}\mathcal{A}$ expresses any element of $\mathcal{A}$ modulo $\mathfrak{q}^n\mathcal{A}$ as $\sum_j b_{j,n} v_j$, with $b_{j,n} \in \mathcal{B}$ and $b_{j,n+1} - b_{j,n} \in \mathfrak{q}^n$. These coefficients converge in $\mathcal{A}$, and their limits lie in $\mathcal{B}$, which is closed since the action is continuous. Thus the v_j generate $\mathcal{A}$ over $\mathcal{B}$. The theorem of Eakin and Nagata [9] now shows that $\mathcal{B}$ is Noetherian. Since $\sqrt{(\mathfrak{m} \cap \mathcal{B})\mathcal{A}} = \mathfrak{m}$, Artin–Rees identifies its maximal-ideal topology with the topology induced from $\mathcal{A}$; hence $\mathcal{B}$ is also complete.

Proposition 2.2. *Let $\mathcal{A}$ satisfy the hypotheses of Theorem 1.1, and suppose that $\mathcal{A}$ has mixed characteristic $(0, p)$. There exist a complete discrete valuation ring C with uniformizer p, with algebraically closed residue field k_0, an integral smooth finite-type C-algebra R, and a regular finite free R-algebra A with R-linear G-action, having the following properties. For some closed k_0-point b of $\operatorname{Spec} R$, the fiber of $X = \operatorname{Spec} A$ over b is supported at a single point x with residue field k_0. For $1 \neq g \in G$, the fixed scheme of g is supported at x, and*

$$\operatorname{length}_{\mathcal{O}_{X,x}}\bigl(\mathcal{O}_{X,x}/I_g\bigr) = i_{\mathcal{A}}(g).$$

Proof. We recall the argument used by Kato and Takeshi Saito [15, proof of Lemma 7.2.10], following the method of Kato, Shuji Saito and Takeshi Saito [13, Lemma 5.3].

By the preceding argument we may assume that $\mathcal{A}$ is complete and finite over the complete Noetherian local ring $\mathcal{B} = \mathcal{A}^G$. Put $d = \dim \mathcal{A} = \dim \mathcal{B}$.

Choose a Cohen coefficient ring $C \subset \mathcal{B}$. It is a complete discrete valuation ring with uniformizer p and residue field k. Choose $u_1, \dots, u_{d-1} \in \mathcal{B}$ so that $p, u_1, \dots, u_{d-1}$ is a system of parameters. The induced map

$$\widehat{R} = C[[u_1, \dots, u_{d-1}]] \longrightarrow \mathcal{B}$$

is finite by complete Nakayama and hence injective by dimension. The displayed parameters form a regular sequence on $\mathcal{A}$, since $\mathcal{A}$ is Cohen–Macaulay. The Auslander–Buchsbaum formula therefore shows that $\mathcal{A}$ is free over $\widehat{R}$.

Next, choose an algebraic closure $\overline{k}$ and an extension $C \subset C'$ of complete discrete valuation rings with common uniformizer p and residue extension $k \subset \overline{k}$. Put $\widehat{R}' = \widehat{R}\widehat{\otimes}_C C'$. Replacing $(C, k, \widehat{R}, \mathcal{A})$ by $(C', \overline{k}, \widehat{R}', \mathcal{A} \otimes_{\widehat{R}} \widehat{R}')$, we may assume k is algebraically closed.

Write

$$R^h = \bigl(C[u_1, \dots, u_{d-1}]_{(p,u_1,\dots,u_{d-1})}\bigr)^h$$

for the henselisation, and write $\mathfrak{q} = (p, u_1, \dots, u_{d-1}) \subset R^h$. By Artin's approximation theorem [3, Theorem 1.10], for every $N \geq 1$, there exists a finite free R^h-algebra $\mathcal{A}^h$ with G-action and a G-equivariant isomorphism

$$\mathcal{A}^h/\mathfrak{q}^N\mathcal{A}^h \simeq \mathcal{A}/\mathfrak{q}^N\mathcal{A}.$$

The algebra $\mathcal{A}^h$ is local because its closed fiber is local. Since $\mathfrak{q}^N\mathcal{A} \subset \mathfrak{m}^N$, the displayed isomorphism induces an equivariant isomorphism modulo the Nth powers of the maximal ideals. For $N \geq 2$ this gives $\operatorname{edim} \mathcal{A}^h = d = \dim \mathcal{A}^h$, so $\mathcal{A}^h$ is regular.

Take $N \geq 2$ larger than the lengths of $\mathcal{A}/I_g(\mathcal{A})$ for $g \neq 1$. Then, by Lemma 2.1,

$$\mathcal{A}^h/I_g(\mathcal{A}^h) \simeq \mathcal{A}/I_g(\mathcal{A}) \qquad (g \neq 1).$$

Finally, descending to a suitable affine étale neighborhood of $C[u_1, \dots, u_{d-1}]$ produces the C-algebra R and the R-algebra A of the proposition. □

3 A parameter lemma

The following lemma is a special case of Spivakovsky's strengthened form of Néron–Popescu desingularization [21, Theorem 1.2]. We give an elementary proof for the hypotheses needed here. Its Jacobian and adjugate calculations follow the method of Elkik [10, §0.2 and I, Lemma 1]; the related implicit-function argument is attributed by Elkik to Tougeron [10, II, Lemma 2].

Lemma 3.1. *Let R be a regular Noetherian domain, Z an affine scheme of finite presentation over R, and $z : \operatorname{Spec} R \to Z$ a section. Write $\operatorname{Sm}(Z/R)$ for the smooth locus of Z/R and put $U = z^{-1}(\operatorname{Sm}(Z/R))$. After replacing $\operatorname{Spec} R$ by a neighborhood of a prescribed point, there are a smooth affine R-scheme T, a section $e : \operatorname{Spec} R \to T$, and an R-morphism $f : T \to Z$ such that*

$$f \circ e = z, \qquad f \text{ is smooth at } e(u) \text{ for every } u \in U.$$

Proof. For a smooth affine integral R-scheme T, a section $e : \operatorname{Spec} R \to T$, and an R-morphism $f : T \to Z$ with $f \circ e = z$, put

$$U(f) = \{u \in U \mid f \text{ is smooth at } e(u)\}, \qquad B(f) = U \setminus U(f).$$

Thus $U(f)$ is open and $B(f)$ is closed in U. We claim that for every $u_0 \in B(f)$ there are a smooth affine morphism $p : T_1 \to T$ with T_1 integral, a section $s : T \to T_1$ of p, and an R-morphism $f_1 : T_1 \to Z$ such that, with $e_1 = s \circ e$, one has

$$f_1 \circ s = f, \qquad U(f) \cup \{u_0\} \subseteq U(f_1).$$

Here $U(f_1)$ is defined using the section e_1.

The lemma follows from this claim by Noetherian induction. Start with $T = \operatorname{Spec} R$, $e = \mathrm{id}$ and $f = z$. Whenever $B(f)$ is nonempty, the claim replaces it by the proper closed subset $B(f_1) \subsetneq B(f)$. Since U is Noetherian, this process terminates with $B(f)$ empty.

We prove the claim, beginning with the preservation of $U(f)$. Suppose that $p : T_1 \to T$ is smooth, $s : T \to T_1$ is a section of p, and $f_1 : T_1 \to Z$ is an R-morphism with $f_1 \circ s = f$. For $u \in U(f)$, write T_u, $T_{1,u}$ and Z_u for the fibers of T, T_1 and Z over $u \in \operatorname{Spec} R$. Every tangent vector to Z_u at $z(u)$ lifts under f to a tangent vector to T_u at $e(u)$. Applying s gives a lift under f_1 to $T_{1,u}$ at $e_1(u)$. Since T_1 and Z are smooth over R at these points, f_1 is smooth at $e_1(u)$.

It remains to construct p, s, f_1 with $f_1 \circ s = f$ and $u_0 \in U(f_1)$. Put $A = \Gamma(T, \mathcal{O}_T)$, a regular Noetherian domain, and set

$$W = Z \times_R T, \qquad \sigma = (f, \mathrm{id}) : T \longrightarrow W, \qquad t_0 = e(u_0).$$

The morphism $W \to T$ is smooth at $\sigma(t_0)$, since $u_0 \in U$. We will construct p, s and a T-morphism $h : T_1 \to W$ with $h \circ s = \sigma$ and h smooth at $s(t_0)$. The projection $W \to Z$ is smooth since T/R is smooth, so composing h with it will give the required f_1.

Choose a presentation $W = \operatorname{Spec}(A[Y_1, \ldots, Y_m]/I)$ and put $c = (\sigma^* Y_1, \ldots, \sigma^* Y_m) \in A^m$. Choose equations $F = (F_1, \ldots, F_q) \subset I$ which generate I near $\sigma(t_0)$, and reorder the variables so that the Jacobian minor

$$\Delta = \det \left(\frac{\partial F_i}{\partial Y_j} \right)_{1 \le i,j \le q}$$

is invertible at $\sigma(t_0)$. Such choices exist by the Jacobian criterion. Clearing denominators gives $a \in ((F) : I)$ with $a(c)$ invertible at t_0; in particular, $a(c)\Delta(c) \neq 0$.

Put $b = a(c)$ and $d = \Delta(c)$, and write the Jacobian matrix of F at c in blocks $(J \mid J_1)$, where

$$J = \left(\frac{\partial F_i}{\partial Y_j}(c) \right)_{1 \le i,j \le q}, \qquad J_1 = \left(\frac{\partial F_i}{\partial Y_j}(c) \right)_{\substack{1 \le i \le q \\ q+1 \le j \le m}}.$$

Thus $\det J = d$. Let J^{ad} be the adjugate of J. For vectors of variables w of length q and v of length $m - q$, define the vector $\delta = (\delta_1, \ldots, \delta_m)$ in two blocks by

$$\delta_{1,\ldots,q} = db^2 J^{\mathrm{ad}} w, \qquad \delta_{q+1,\ldots,m} = d^2 b^2 v. \tag{3.1.1}$$

Since $F(c) = 0$, expansion at c gives

$$\begin{aligned} F(c + \delta) &= J\delta_{1,\ldots,q} + J_1 \delta_{q+1,\ldots,m} + d^2 b^4 Q(w, v) \\ &= db^2 J J^{\mathrm{ad}} w + d^2 b^2 J_1 v + d^2 b^4 Q(w, v) \\ &= d^2 b^2 \big(w + J_1 v + b^2 Q(w, v) \big). \end{aligned} \tag{3.1.2}$$

Here $Q \in A[w, v]^q$ has no terms of degree less than two: each term beyond the linear part contains at least two factors δ_i, each divisible by db^2. The last equality uses $JJ^{\mathrm{ad}} = d\,\mathrm{id}$. Consider the closed subscheme of $\mathbb{A}^m_A = \operatorname{Spec} A[w, v]$ defined by

$$w + J_1 v + b^2 Q(w, v) = 0.$$

Along the section $w = v = 0$, the Jacobian matrix with respect to w is id_q. Inverting its determinant gives an affine open neighborhood T_1 of the image of this section in the subscheme just defined. The projection $(w, v) \mapsto v$ defines an étale morphism $T_1 \to \mathbb{A}^{m-q}_A$, and hence a smooth morphism $p : T_1 \to T$. The section $s : T \to T_1$ is given by $w = v = 0$.

Furthermore,

$$a(c+\delta) = b(1+bQ')$$

for a polynomial $Q' \in A[w,v]$ vanishing at zero. Invert $1+bQ'$. Since T_1 is flat over the domain A, multiplication by b is injective on $\mathcal{O}_{T_1}$. For $P \in I$, the relation $aP \in (F)$ and (3.1.2) now give $P(c+\delta) = 0$. Thus $c+\delta$ defines a T-morphism $h : T_1 \to W$ with $h \circ s = \sigma$. Since T_1 is smooth over the regular ring A, it is regular. The connected image $s(T)$ lies in a single connected component of T_1, which is open, closed and integral. Replacing T_1 by this component keeps it affine and smooth over T, and makes it integral.

To check smoothness of h, fix $t \in D(bd) \subset T$ and write $T_{1,t}$ and W_t for the fibers of $p : T_1 \to T$ and $W \to T$ over t. In the following calculation, all matrices and scalars are taken over $\kappa(t)$. Since Q has no linear terms, a tangent vector $(\dot{w}, \dot{v})$ to $T_{1,t}$ at $s(t)$ satisfies $\dot{w} + J_1\dot{v} = 0$. Thus these tangent vectors have the form $(-J_1\tau, \tau)$, with $\tau \in \kappa(t)^{m-q}$. By (3.1.1), h sends such a tangent vector to

$$\big(-db^2 J^{\mathrm{ad}} J_1\tau,\ d^2b^2\tau\big) = (bd)^2\big(-J^{-1}J_1\tau,\ \tau\big).$$

Since $a(c) = b$ is invertible at t, the F_i generate I near $\sigma(t)$. Hence the tangent vectors (ξ, η) to W_t at $\sigma(t)$ are exactly the solutions of $J\xi + J_1\eta = 0$. As $\det J = d$ is invertible, they are precisely $(-J^{-1}J_1\tau, \tau)$ for $\tau \in \kappa(t)^{m-q}$. The scalar $(bd)^2$ is also invertible, so h induces an isomorphism between the tangent space of $T_{1,t}$ at $s(t)$ and that of W_t at $\sigma(t)$. Both T_1 and W are smooth over T at these points, so h is étale at $s(t)$. Since $t_0 \in D(bd)$ and $W \to Z$ is smooth, the composite $f_1 : T_1 \xrightarrow{h} W \to Z$ is smooth at $e_1(u_0) = s(t_0)$. It satisfies $f_1 \circ s = f$, and hence $U(f) \cup \{u_0\} \subseteq U(f_1)$, proving the claim. □

4 Equivariant deformation

We now work over the ring C constructed in Proposition 2.2, and write k for its algebraically closed residue field k_0. All group actions on C-schemes are over C.

Finite algebras with group action

For a positive integer n, let $Z_{n,G}$ be the scheme of commutative associative algebra structures with unit e_1 on a free module with ordered basis $e_1, \ldots, e_n$, together with an action of G by algebra automorphisms. Thus, for a ring B, a B-point is an algebra structure on B^n with specified unit e_1, and invertible matrices ρ_g preserving multiplication and the unit, with $\rho_1 = 1$ and $\rho_{gh} = \rho_g\rho_h$. This defines an affine scheme of finite type over $\mathbf{Z}$. Without the G-action, this is the scheme of based algebras with specified first basis vector as unit studied by Poonen [18, Proposition 1.1 and Definition 1.5]. Denote the spectrum of the universal algebra by

$$W_{n,G} \longrightarrow Z_{n,G}.$$

If a B-point $z \in Z_{n,G}(B)$ corresponds to a B-algebra D, then lifting z to a B-point of $W_{n,G}$ is equivalent to giving an augmentation $D \to B$. We will use this functor of points description of $W_{n,G}$.

Lemma 4.1. *Let z be a geometric point of $Z_{n,G}$ corresponding to a finite algebra which is a local complete intersection and has a free G-action. Then $Z_{n,G}$ is smooth over $\mathbf{Z}$ at z, and $W_{n,G}$ is smooth over $\mathbf{Z}$ at every point above z.*

Proof. When G is trivial, these assertions are the smoothness of the finite syntomic parameter space and of its universal family, as proved in The Stacks Project [22, Lemma 49.11.4 and Remark 49.11.6]. For $W_{n,G}$, one needs to check that one can lift the augmentation corresponding to a point of $W_{n,G}$. The complete-intersection presentation can be chosen with its variables in the augmentation ideal; lifting its equations with zero constant term lifts both the algebra and the augmentation.

For general G, let $B' \twoheadrightarrow B$ be a small extension of local Artinian rings, and let D/B be the G-equivariant B-algebra corresponding to a point $z_B \in Z_{n,G}(B)$ lifting z. It is finite syntomic over B, since it is finite flat and its geometric closed fiber is a complete intersection; see The Stacks Project [22, Lemma 49.11.1]. As G acts freely on $\operatorname{Spec} D$, $\operatorname{Spec} D \to \operatorname{Spec} D^G$ is a finite étale G-torsor. Hence the algebra D^G/B is finite syntomic by faithfully flat descent of flatness and étale descent of the complete-intersection property. By the case when G is trivial, it lifts to $(D^G)'/B'$. Grothendieck's invariance theorem under nilpotent thickenings [11, Exposé I, Theorem 8.3] lifts the finite étale torsor $\operatorname{Spec} D \to \operatorname{Spec} D^G$ and its action uniquely to a finite étale G-torsor $\operatorname{Spec} D' \to \operatorname{Spec}(D^G)'$.

Finally, if $\tilde{z}_B \in W_{n,G}(B)$ lifts z_B, then we may also lift the corresponding augmentation $D \to B$, by first choosing the lift $(D^G)'/B'$ together with an augmentation $(D^G)' \to B'$ lifting $D^G \to B$, and then using formal étaleness to extend it to $D' \to B'$ lifting $D \to B$. Lifting the chosen basis of D/B to D'/B', with first vector 1, completes the proof by the infinitesimal criterion for smoothness. □

Invariant sections

The transversality argument below is the classical Bertini incidence argument, applied on the free quotient; compare Katz [8, Exposé XVII, §3.1]. The congruences imposed on the sections will also preserve the fixed algebras.

We shall shrink affine G-schemes about a fixed point x. Such shrinkings may be taken affine and invariant: replace a principal neighborhood $D(a)$ by $D(\prod_{g\in G} g(a))$.

For a C-scheme Y, write $Y_s = Y \times_C k$ for its special fiber and $\operatorname{Crit}(Y/C)$ for its reduced nonsmooth locus.

Lemma 4.2. *Let $Y = \operatorname{Spec} D$ be regular, flat, and of finite type over C. Let G act on Y, fixing a closed k-point x. Suppose that invariant functions $t_1, \dots, t_r$ define a subscheme X which is regular of codimension r and flat over C near x. Assume that*

$$X \cap \operatorname{Crit}(Y/C) \quad \text{and} \quad \operatorname{Fix}_X(g) \ (g \neq 1)$$

are supported at x. For every integer $N \geq 2$, there are invariant functions $t_i' = t_i + \delta_i$, with $\delta_i \in \mathfrak{m}_x^N$, whose common zero scheme X' has the following properties after an invariant shrinking about x:

(i) X' is regular and flat over C, and $\operatorname{Crit}(X'/C)$ is supported at x;

(ii) $\mathcal{O}_{X',x}/I_g(X')_x \simeq \mathcal{O}_{X,x}/I_g(X)_x$ for every $g \neq 1$;

(iii) the order-N truncations at x of X' and X are G-equivariantly isomorphic.

Proof. Restrict to the component of Y containing x, and let $\mathfrak{n} \subset D$ be the ideal of x. The case $r = 0$ is immediate. Fix $N \geq 2$. Let H define $\operatorname{Crit}(Y/C)$. Set

$$J = \bigcap_{g\neq 1} I_g(D), \qquad \mathfrak{a} = HJ\mathfrak{n}^N.$$

We shall choose $\delta_i \in \mathfrak{a}^G$.

Put $U = Y \setminus \big(V(H) \cup V(J) \cup \{x\}\big)$. The quotient

$$U \longrightarrow V = U/G$$

is a finite étale torsor; see The Stacks Project [22, Lemma 66.14.2]. Thus V is smooth over C, and is open in $\operatorname{Spec} D^G$. The orbit polynomials show that D is finite over D^G; hence D^G is of finite type over C by Artin–Tate. The ideal $\mathfrak{a}$ restricts to the unit ideal on U. For each orbit in U, prime avoidance gives an element of $\mathfrak{a}$ nonzero along the orbit. Taking its norm and using quasi-compactness gives finitely many $a_j \in \mathfrak{a}^G$ with no common zero on V.

We choose the perturbations first on the special fiber V_s. The invariant functions t_i descend to V; write $\bar{t}_i$ for their restrictions to V_s. We seek perturbations $\bar{\delta}_i$ for which the common zero scheme of $\bar{t}_1 + \bar{\delta}_1, \dots, \bar{t}_r + \bar{\delta}_r$ is smooth of codimension r in V_s. All differentials in this construction are taken on V_s over k. If V_s is empty, take $\delta_i = 0$ and omit this choice. Otherwise, V_s is smooth of pure dimension

$$n = \dim \mathcal{O}_{X,x} + r - 1 \geq r.$$

The equality follows from the dimension formula for Y/C, and the inequality from the flatness of X/C.

Choose generators b_h of the C-algebra D^G, and let E be the finite-dimensional k-space of functions on V_s spanned by the restrictions of the a_j and $a_j b_h$. At every closed point $v \in V_s$, these functions allow us to prescribe both a value and a differential: the map

$$E \longrightarrow k \oplus T_v^* V_s, \qquad \epsilon \longmapsto \big(\epsilon(v), d_v \epsilon\big)$$

is surjective. Here $T_v^* V_s$ is the cotangent space of V_s at v. Indeed, choose j with $a_j(v) \neq 0$. Since the b_h generate D^G over C, their differentials $d_v b_h$ span $T_v^* V_s$. The functions

$$a_j(b_h - b_h(v)) = a_j b_h - b_h(v) a_j$$

belong to E, vanish at v, and have differentials $a_j(v) d_v b_h$. They therefore give every differential with value zero, while a_j supplies an arbitrary value.

Now vary a tuple $\boldsymbol{\epsilon} = (\epsilon_1, \dots, \epsilon_r)$ in the affine space E^r. Let $\Sigma \subset V_s \times E^r$ be the closed locus of pairs $(v, \boldsymbol{\epsilon})$ for which all the functions $\bar{t}_i + \epsilon_i$ vanish at v and their differentials in $T_v^* V_s$ are linearly dependent. For a fixed v, the surjectivity just proved shows that vanishing imposes r independent conditions on the tuple, and that, subject to these conditions, its r differentials can be prescribed arbitrarily. In a basis of $T_v^* V_s$ they form an arbitrary r by n matrix. The locus where its rank is less than r has codimension $n - r + 1$. Thus the fiber of $\Sigma \to V_s$ over v has codimension $n + 1$ in E^r, and

$$\dim \Sigma \leq n + \dim E^r - (n + 1) = \dim E^r - 1.$$

The closure of the image of Σ in E^r is therefore a proper subset. Since k is infinite, choose a k-point outside this closure, and write $\bar{\delta}_i$ for its components. At every common zero of the $\bar{t}_i + \bar{\delta}_i$, their differentials are independent. The Jacobian criterion shows that their common zero scheme in V_s is smooth of codimension r.

Express each $\bar{\delta}_i$ as a k-linear combination of the restrictions of the a_j and $a_j b_h$, and lift the coefficients to C. This gives $\delta_i \in \mathfrak{a}^G$ lifting $\bar{\delta}_i$. We set $X' \subset Y$ to be the subscheme defined by $t_i' = t_i + \delta_i$. The relative differentials of the t_i' are linearly independent along $X_s' \cap U_s$, since their reductions are pulled back along the finite étale map $U_s \to V_s$ from the independent differentials just constructed. The relative Jacobian criterion therefore shows that X'/C is smooth along this locus.

Since $\delta_i \in H \cap I_g(D)$ for every $g \neq 1$, one has

$$\begin{aligned} D/(t_1', \dots, t_r', I_g(D)) &= D/(t_1, \dots, t_r, I_g(D)), \\ D/(t_1', \dots, t_r', H) &= D/(t_1, \dots, t_r, H). \end{aligned} \tag{4.2.1}$$

These equalities show that $X' \setminus \{x\}$ lies in U. Thus X'/C is smooth along $X'_s \setminus \{x\}$.

Finally we show that $\mathrm{Crit}(X'/C)$ is supported at x. The images of the t_i in $\mathfrak{m}_{Y,x}/\mathfrak{m}_{Y,x}^2$ are linearly independent, since $\mathcal{O}_{X,x}$ is regular of codimension r. As $\delta_i \in \mathfrak{n}^N$, the t_i' are also linearly independent, so $\mathcal{O}_{X',x}$ is regular. The component of X' through x has positive dimension, since $\dim \mathcal{O}_{X',x} = \dim \mathcal{O}_{X,x} \geq 1$. It cannot lie in the special fiber, as this would contradict smoothness over C at the generic point of this component. Hence a uniformizer of C is nonzero in the regular local domain $\mathcal{O}_{X',x}$, so X' is flat over C at x. After an invariant affine shrinking, X' is regular and flat. Its generic fiber is smooth in characteristic zero, and its special fiber is smooth away from x. Together with (4.2.1) and the equality of the order-N truncations, this proves the lemma. □

The deformation proposition

Proposition 4.3. *Let R be an integral smooth finite-type C-algebra, and let A be a regular finite free R-algebra with R-linear G-action. Suppose that the fiber of $X = \operatorname{Spec} A$ over a closed k-point b of $\operatorname{Spec} R$ is supported at a single point x, and that for $1 \neq g \in G$, the fixed scheme of g is supported at x. There is a regular flat affine finite-type G-scheme X'/C, with a closed k-point x', such that*

$$\mathrm{Crit}(X'/C),\ \mathrm{Fix}_{X'}(g)\ (g \neq 1) \quad \textit{are supported at } x',$$

and

$$\mathcal{O}_{X',x'}/I_g(X')_{x'} \simeq \mathcal{O}_{X,x}/I_g(X)_x \qquad (g \neq 1).$$

For any $N \geq 1$, we may also arrange that $\mathcal{O}_{X',x'}/\mathfrak{m}_{x'}^N \simeq \mathcal{O}_{X,x}/\mathfrak{m}_x^N$.

Proof. We will embed X, near x, in a regular flat C-scheme Y by invariant equations, with Y/C smooth along $X \setminus \{x\}$. Lemma 4.2 will then give the required deformation.

After shrinking $\operatorname{Spec} R$ about b, choose an R-basis of A beginning with 1. The point x is fixed by G, since it is the unique point of its fiber, and its residue field is k, since k is algebraically closed. The multiplication and G-action on A, in the chosen basis, define a section

$$z : \operatorname{Spec} R \longrightarrow Z_R = Z_{n,G} \times_{\mathbf{Z}} \operatorname{Spec} R, \qquad n = \operatorname{rank}_R A.$$

Write $W_R = W_{n,G} \times_{\mathbf{Z}} \operatorname{Spec} R$ for the universal family over Z_R. Its pullback along z is X.

The finite flat morphism of regular schemes $X \to \operatorname{Spec} R$ is a local complete intersection, and away from b the G-action on X is free. Lemma 4.1 therefore shows that Z_R/R is smooth along $z(\operatorname{Spec} R \setminus \{b\})$, and W_R/R is smooth above this locus.

Apply Lemma 3.1 at b. It gives a smooth affine R-scheme T, a section $e : \operatorname{Spec} R \to T$, and an R-morphism $f : T \to Z_R$ such that $f \circ e = z$ and f is smooth along $e(\operatorname{Spec} R \setminus \{b\})$. Put $Y = W_R \times_{Z_R} T$. Then $Y \times_{T,e} \operatorname{Spec} R = X$. The projection $Y \to T$ is finite flat, and the universal G-action on Y is trivial on T. Thus Y is flat over C. At a point of $X \setminus \{x\}$, the projection $Y \to W_R$ is smooth, being the base change of f, and W_R/R is smooth at its image. Since R/C is smooth, Y/C is smooth along $X \setminus \{x\}$.

We next choose the invariant equations defining X in Y. Let r be the relative dimension of T/R at $e(b)$. As T is smooth over R, after shrinking $\operatorname{Spec} R$ about b and T about $e(b)$, there is

an étale map $T \to \mathbb{A}^r_R$ for which $e(\operatorname{Spec} R)$ is the inverse image of the zero section. We denote by $t_1, \dots, t_r$ the pullbacks to T of the coordinate functions. Pulling back to Y gives

$$X = V(t_1, \dots, t_r) \subset Y.$$

The t_i form a regular sequence on T, and hence on Y by flatness. Their pullbacks are invariant because G acts trivially on T. The quotient $\mathcal{O}_{Y,x}/(t_1, \dots, t_r) = \mathcal{O}_{X,x}$ is regular, so $\mathcal{O}_{Y,x}$ is regular as well. Shrink Y to an invariant regular affine neighborhood of x, and replace X by its intersection with this neighborhood.

We have now obtained all the hypotheses of Lemma 4.2: X is regular of codimension r in Y and flat over C, while $X \cap \operatorname{Crit}(Y/C)$ and the nonidentity fixed schemes of X are supported at x. Applying that lemma gives our result. □

5 The trace formula and the character

Below we consider a pointed C-scheme (X, x) with G-action, satisfying the conclusion of Proposition 4.3. That is, X is a regular, C-flat, affine G-scheme, which is smooth outside the k-point x, and such that $\operatorname{Fix}_X(g)$ is supported on x for $g \neq 1$. Below we use only that the singular locus of X/C is proper over C.

We first express the fixed-point multiplicities as intersection numbers. The results of Beraldo and Pippi then express these numbers as traces on two fixed virtual representations. An elementary character argument will give the prescribed value at 1 and prove positivity.

The local intersection

We refer the reader to Bloch's conductor conjecture [7] and the localized intersection product of Kato and Takeshi Saito [14, Theorem 3.2.1 and Definition 3.2.2] for background.

Put $P = X \times_C X$, let $\Delta \subset P$ be the diagonal, and let $\Gamma_g \subset P$ be the image of $(1, g) : X \to P$. For $g \neq 1$ and integers $i \geq 0$, the sheaves

$$\mathcal{T}_i = \mathcal{T}or_i^{\mathcal{O}_P}(\mathcal{O}_\Delta, \mathcal{O}_{\Gamma_g})$$

are supported at (x, x), since $\Delta \cap \Gamma_g \simeq \operatorname{Fix}_X(g)$, and their stalks there have finite length over C. The stable intersection number is defined as

$$[[\mathcal{O}_\Delta, \mathcal{O}_{\Gamma_g}]]_C = \operatorname{length}_C(\mathcal{T}_{2q}) - \operatorname{length}_C(\mathcal{T}_{2q+1}), \qquad q \gg 0. \tag{5.1}$$

The result below calculates the two lengths on the right-hand side and shows that the left-hand side is the multiplicity of the fixed point of g.

Lemma 5.1. *For $g \neq 1$ and $n \geq 0$, one has*

$$\mathcal{T}_n \simeq \begin{cases} \mathcal{O}_{X,x}/I_g(X)_x, & n \text{ even}, \\ 0, & n \text{ odd}. \end{cases}$$

In particular,

$$[[\mathcal{O}_\Delta, \mathcal{O}_{\Gamma_g}]]_C = \operatorname{length}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/I_g(X)_x).$$

Proof. The equality of intersection numbers follows from Kato and Takeshi Saito's comparison theorem [14, Proposition 3.2.3], using their local hypersurface embedding [14, Lemma 5.1.1]. We give a direct local proof.

Fix $g \neq 1$. We will compute the Tor groups $\mathcal{T}_n$ using a hypersurface presentation of the completed local ring. Put $\mathcal{B} = \widehat{\mathcal{O}}_{X,x}$ and $d = \dim \mathcal{B}$. A uniformizer π of C cannot be part of a regular system of parameters of $\mathcal{B}$, as then $I_g(\mathcal{B})$ would be generated by $d-1$ elements, contradicting that $\mathcal{B}/I_g(\mathcal{B})$ has finite length. Choose regular parameters $x_1, \dots, x_d$ of $\mathcal{B}$. The resulting surjection $C[[x_1, \dots, x_d]] \to \mathcal{B}$ has a height-one prime kernel, which is principal since the source is regular. Thus

$$\mathcal{B} = C[[x_1, \dots, x_d]]/(F_0).$$

To resolve the diagonal, use the completed local ring

$$\mathcal{D} = \widehat{\mathcal{O}}_{P,(x,x)} = \mathcal{B}[[y_1, \dots, y_d]]/(F_0(y)),$$

where the x_i and y_i are the parameters from the first and second factors. Put $Q = \mathcal{B}[[y_1, \dots, y_d]]$ and $v_i = y_i - x_i \in Q$. Since $F_0(x) = 0$ in $\mathcal{B}$, we may write $F_0(y) = \sum_i c_i v_i$ with $c_i \in Q$. We use the same letters for the images of the c_i in $\mathcal{D}$. We have $\mathcal{D}/(v_1, \dots, v_d) = \mathcal{B}$. Let K_Q denote the Koszul complex on the v_i, and $K_{\mathcal{D}}$ the Koszul complex on the images of the v_i in $\mathcal{D}$. Thus,

$$K_{\mathcal{D}} = \bigwedge\nolimits^{\bullet}_{\mathcal{D}}(\mathcal{D}e_1 \oplus \cdots \oplus \mathcal{D}e_d), \qquad |e_i| = 1, \qquad de_i = v_i,$$

and similarly for K_Q. As $v_1, \dots, v_d$ is a regular sequence in Q, K_Q is a resolution of

$$Q/(v_1, \dots, v_d) \simeq \mathcal{B} \simeq \mathcal{D}/(v_1, \dots, v_d).$$

Since $F_0(y)$ is a nonzero divisor in Q, the exact sequence of Koszul complexes obtained from

$$0 \longrightarrow Q \xrightarrow{F_0(y)} Q \longrightarrow \mathcal{D} \longrightarrow 0$$

shows that $H_0(K_{\mathcal{D}}) = H_1(K_{\mathcal{D}}) = \mathcal{B}$, with all other homology zero. The element $z = \sum_i c_i e_i \in K_Q$ maps to a cycle in $K_{\mathcal{D}}$, whose class generates the $\mathcal{B}$-module $H_1(K_{\mathcal{D}})$: the connecting map sends this class to $1 \in \mathcal{B}$, since $dz = F_0(y)$.

To construct a resolution of $\mathcal{B}$ by free $\mathcal{D}$-modules, we use a construction of Tate [23]. Let $\Gamma_{\mathcal{D}}(t) = \bigoplus_{j \geq 0} \mathcal{D}t^{[j]}$ be the divided power algebra, graded with $t^{[j]}$ in degree $2j$. Consider the complex

$$K_{\mathcal{D}} \otimes_{\mathcal{D}} \Gamma_{\mathcal{D}}(t), \qquad |t| = 2, \qquad dt = \sum_i c_i e_i, \tag{5.1.1}$$

Its term in homological degree i is the degree-i homogeneous component of the tensor product. The differential extends the Koszul differential by the graded Leibniz rule and

$$d(t^{[j]}) = \Big(\sum_i c_i e_i\Big) t^{[j-1]} \quad (j \geq 1).$$

This differential squares to zero because z is a Koszul cycle and $z \wedge z = 0$. We verify that this complex resolves $\mathcal{B}$ over $\mathcal{D}$. Filter (5.1.1) by divided power degree, and consider the associated spectral sequence. The differential on the associated graded is the Koszul differential, since the additional term lowers divided power degree by one. After taking Koszul homology, the differential on the E_1-page is induced by $dt = z$ and gives an isomorphism

$$H_0(K_{\mathcal{D}})t^{[j]} \simeq \mathcal{B}t^{[j]} \xrightarrow{\sim} \mathcal{B} \cdot [z]t^{[j-1]} \simeq H_1(K_{\mathcal{D}})t^{[j-1]} \qquad (j \geq 1).$$

Thus, on the E_2-page only $\mathcal{B}t^{[0]}$ in degree zero remains. The filtration is finite in each total degree, so the spectral sequence converges and shows that (5.1.1) resolves $\mathcal{B}$ as a $\mathcal{D}$-module.

We now tensor this resolution with $\mathcal{D}/(y_1 - g(x_1), \dots, y_d - g(x_d)) \simeq \mathcal{B}$ to compute the $\mathcal{T}_n$. We obtain the complex $K_{\mathcal{B}} \otimes_{\mathcal{B}} \Gamma_{\mathcal{B}}(t)$, where $K_{\mathcal{B}}$ is the Koszul complex on $g(x_1) - x_1, \dots, g(x_d) - x_d \in \mathcal{B}$. Since $\mathcal{B}/I_g(\mathcal{B})$ has finite length, they form a system of parameters of the regular ring $\mathcal{B}$, and hence a regular sequence. Thus $K_{\mathcal{B}}$ is a resolution of $\mathcal{B}/I_g(\mathcal{B})$. Using the filtration by divided power degree, we again obtain a spectral sequence whose E_1-page is

$$(\mathcal{B}/I_g(\mathcal{B})) \otimes_{\mathcal{B}} \Gamma_{\mathcal{B}}(t),$$

a graded module concentrated in even total degrees. Thus the spectral sequence degenerates at the E_1-page, the even Tor groups are $\mathcal{B}/I_g(\mathcal{B})$ and the odd Tor groups are zero. □

The inputs from Beraldo and Pippi

Blanc, Robalo, Toën and Vezzosi identified the realization of a singularity category with inertia-invariant vanishing cycles in the proper flat case [6, Theorem 4.39]. Toën and Vezzosi used this comparison to obtain a categorical form of Bloch's formula, which relates the localized self-intersection of the diagonal to the Euler characteristic of vanishing cycles together with its Swan conductor [24, Theorem 5.2.2 and Remark 5.2.3]. We need a version of this result which computes the intersection of the diagonal with the graph of an element $g \in G$, for which we use the results of Beraldo and Pippi [4]. We recall some results from *loc. cit*, adapted to our setting, and then use these to prove Lemma 5.2 below.

Fix a prime $\ell \neq \operatorname{char} k$. Put $K = \operatorname{Frac}(C)$, choose an algebraic closure $\overline{K}$ of K, and put $I_K = \operatorname{Gal}(\overline{K}/K)$, which is the inertia group since k is algebraically closed. We use Deligne's normalization of the vanishing-cycle complex [8, Exposé XIII, §2.1]

$$\mathbf{Q}_{\ell X_s} \longrightarrow R\Psi_{X/C}\mathbf{Q}_\ell \longrightarrow R\Phi_{X/C}\mathbf{Q}_\ell \longrightarrow \mathbf{Q}_{\ell X_s}[1].$$

Let $\beta \in \mathbf{Q}_\ell(1)$ be a basis vector, and set

$$\nu_0 = R\Gamma(X_s, R\Phi_{X/C}\mathbf{Q}_\ell), \qquad \Lambda = \mathbf{Q}_\ell[\beta, \beta^{-1}] = \bigoplus_{n \in \mathbf{Z}} \mathbf{Q}_\ell(n)[2n], \qquad \nu = \nu_0 \otimes_{\mathbf{Q}_\ell} \Lambda,$$

so β has degree -2.

Let $F = R\Gamma(I_K, \Lambda)$, with its natural augmentation $F \to \Lambda$, so that the cohomology of F is two-periodic up to Tate twist. The vanishing-cycle sheaves are constructible and supported on the proper locus $\operatorname{Crit}(X/C)$, so ν_0 is bounded with finite-dimensional cohomology. Its G-action is $g \mapsto (g^{-1})^*$ and commutes with inertia. All tensor products below are derived, and superscripts I_K and I_L denote derived continuous invariants.

For a nontrivial finite Galois extension L/K in $\overline{K}$, put $I_L = \operatorname{Gal}(\overline{K}/L)$, $H = I_K/I_L$, and

$$\begin{gathered} R_0 = F, \qquad R_1 = F[H](1 - e_0), \qquad e_0 = \frac{1}{|H|}\sum_{h \in H} h, \\ M_0 = \nu^{I_K}[-1], \qquad M_1 = \operatorname{cofib}\big(\nu^{I_K} \longrightarrow \nu^{I_L}\big). \end{gathered} \tag{5.2}$$

Restriction also defines the F-action on ν^{I_L}. The group H acts on ν^{I_L}, and we may consider the projections corresponding to the idempotents e_0 and $1 - e_0$.

$$\nu^{I_K} \simeq e_0\nu^{I_L}, \qquad M_1 \simeq (1 - e_0)\nu^{I_L}.$$

Thus M_1 is a module over the reduced group algebra R_1. The G-actions on M_0 and M_1 commute with these coefficient actions. Both modules, and the extension L/K, will be chosen once for the whole group G.

Choose L/K sufficiently large that I_L acts unipotently on the cohomology sheaves of $R\Phi_{X/C}\mathbf{Q}_\ell$, as in the proof of [5, Theorem 8.4.3]. Beraldo and Pippi identify the coefficient algebras and modules in (5.2), and prove that M_i is perfect over R_i [4, Proposition 5.4.1 and Propositions 5.5.2–5.5.5]. Both modules are also perfect over F, since R_1 is a direct summand of the finite free F-module $F[H]$.

Recall also the duality in these results. Verdier duality for inertia-invariant vanishing cycles identifies the right duals of M_0 and M_1 with M_0 and $M_1(1)$, respectively [4, Propositions 5.4.1 and 5.5.5]. For M_1, the comparison of [4, Proposition 5.5.7 and Remark 5.6.4] identifies $M_1 \otimes_{R_1} M_1(1)$ with $M_1 \otimes_{R_1} M_1$. They write the diagonal evaluation as

$$\mathrm{ev}_i^{\mathcal{O}_\Delta} : M_i \otimes_F M_i \longrightarrow R_i,$$

with the first factor carrying the right R_i-module structure. Together with the diagonal coevaluation, this makes the first copy of M_i a dual of the second [4, Corollaries 5.7.10 and 5.7.13]. Under the resulting identification of the first factor with $R\mathrm{Hom}_{R_i}(M_i, R_i)$, evaluation is the pairing $d \otimes m \mapsto d(m)$.

Let C_L be the integral closure of C in L, and let $a_H = a_{C_L,H}$ be the classical Artin character, defined by (1.1). For a finite group H, write $K_0(\mathbf{Q}_\ell[H])$ for its representation ring over $\mathbf{Q}_\ell$.

Lemma 5.2 (The graph trace formula)**.** *Under the hypotheses on X/C and G at the beginning of this section, there exist a finite Galois extension L/K, with group H, and virtual representations*

$$V_0 \in K_0(\mathbf{Q}_\ell[G]), \qquad V_1 \in K_0(\mathbf{Q}_\ell[G \times H]),$$

such that, for every $g \neq 1$,

$$[[\mathcal{O}_\Delta, \mathcal{O}_{\Gamma_g}]]_C = \mathrm{Tr}(g, V_0) - \frac{1}{|H|} \sum_{h \in H} a_H(h)\, \mathrm{Tr}\big((g, h^{-1}), V_1\big). \tag{5.2.1}$$

Proof. Use the extension L/K chosen above, and put

$$U_i = M_i \otimes_F \Lambda, \qquad V_i = [H^0(U_i)] - [H^1(U_i)] \quad (i = 0, 1).$$

As the M_i are perfect over F the U_i are perfect over Λ, so the two cohomology groups are finite-dimensional. They carry the G-action and, for $i = 1$, the commuting H-action. Thus V_0 and V_1 belong to the stated representation rings.

Beraldo and Pippi observe in [4, Remark 1.6.4] that the argument of that paper extends to endomorphisms. We make this explicit, and deduce the resulting formula by using the twisted conductor formula of their other paper [5, §8.5.6], applied to $\mathcal{O}_\Delta$ and $\mathcal{O}_{\Gamma_g}$. Its geometric term is the stable even Tor degree minus the stable odd Tor degree [5, §8.5.2], which is $[[\mathcal{O}_\Delta, \mathcal{O}_{\Gamma_g}]]_C$ since the Tor sheaves are supported at the k-point (x, x). Using [5, Lemma 8.4.4(1)–(2)], the formula gives

$$[[\mathcal{O}_\Delta, \mathcal{O}_{\Gamma_g}]]_C = \mathrm{Tr}_F^{(\mathcal{O}_\Delta, \mathcal{O}_{\Gamma_g})}(M_0) - \alpha_H\big(\mathrm{Tr}_{R_1}^{(\mathcal{O}_\Delta, \mathcal{O}_{\Gamma_g})}(M_1)\big).$$

Let us explain the right side of the formula. It takes values in $H^0(F) \supset \mathbf{Q}_\ell$ (in fact $H^0(F) = \mathbf{Q}_\ell$, see below). The expression $\mathrm{Tr}_{R_i}^{(\mathcal{O}_\Delta, \mathcal{O}_{\Gamma_g})}(M_i)$ is the twisted trace defined in [5, §8.5.5]. It takes values in $\mathrm{HH}_0(R_i/F) = H^0(\mathrm{HH}(R_i/F))$, where the relative Hochschild homology complex is

$$\mathrm{HH}(R_i/F) = R_i \otimes_{R_i^e} R_i, \qquad R_i^e = R_i \otimes_F R_i^{\mathrm{op}}.$$

For the graph Γ_g, Beraldo and Pippi's construction gives the twisted trace as the degree-zero class of the composite

$$F \xrightarrow{(1\otimes g)\circ \mathrm{coev}_i^{\mathcal{O}_\Delta}} M_i \otimes_{R_i} M_i \xrightarrow{\sim} (M_i \otimes_F M_i) \otimes_{R_i^e} R_i$$
$$\xrightarrow{\mathrm{ev}_i^{\mathcal{O}_\Delta} \otimes 1} R_i \otimes_{R_i^e} R_i = \mathrm{HH}(R_i/F).$$

The first arrow is defined by composing the diagonal coevaluation with $1 \otimes g$, where g acts as $g_* = (g^{-1})^*$. The last arrow is induced by the diagonal evaluation recalled above, as in [5, §§8.5.3–8.5.4]. Here we use $\mathcal{O}_{\Gamma_g} = (1, g)_* \mathcal{O}_\Delta$ and functoriality of the construction of *loc. cit* with respect to pushforward on the second factor [4, Theorem 3.2.8, Corollary 4.3.6 and Proposition 5.6.3]. The diagonal evaluation and coevaluation satisfy the duality identities [4, Corollaries 5.7.10 and 5.7.13].

The Artin functional

$$\alpha_H : \mathrm{HH}_0(R_1/F) \to \mathrm{HH}_0(F/F) \simeq H^0(F)$$

is the $H^0(F)$-linear map which sends the class of $h(1 - e_0)$ to $a_H(h)$. We denote this class by $[h]$. For M_0, the displayed composite gives the scalar F-linear trace:

$$\mathrm{Tr}_F^{(\mathcal{O}_\Delta, \mathcal{O}_{\Gamma_g})}(M_0) = \mathrm{Tr}_F(g, M_0) \in H^0(F).$$

For M_1, Hattori's group-algebra trace calculation [12, §1.2, Example 7], in the form of [4, Lemma A.1.4 and Corollary A.2.2], applies directly to the diagonal evaluation and coevaluation. With g inserted as above, it gives

$$\mathrm{Tr}_{R_1}^{(\mathcal{O}_\Delta, \mathcal{O}_{\Gamma_g})}(M_1) = \frac{1}{|H|} \sum_{h \in H} \mathrm{Tr}_F(gh^{-1}, M_1)[h].$$

Here the traces on the right are again F-linear traces, taking values in $H^0(F)$.

Finally, the augmentation identifies $H^0(F)$ with $H^0(\Lambda) = \mathbf{Q}_\ell$: inertia has ℓ-cohomological dimension one, and its degree-one cohomology contributes only odd degrees after periodization. Extension of scalars preserves traces, so

$$\mathrm{Tr}_F(g, M_i) = \mathrm{Tr}(g, H^0(U_i)) - \mathrm{Tr}(g, H^1(U_i)).$$

The same identity holds for gh^{-1} on M_1. Substituting in the preceding trace formulas proves (5.2.1). □

The virtual representation

The weighted sum in Lemma 5.2 is itself a virtual character. We record the elementary identity that makes this integrality explicit.

Lemma 5.3. *Let H be a finite group, let E be a finite-dimensional $\mathbf{Q}_\ell[H]$-module with character a, and let $U \in K_0(\mathbf{Q}_\ell[G \times H])$. Then $\mathrm{Hom}_H(E, U) \in K_0(\mathbf{Q}_\ell[G])$ is defined by additivity, and*

$$\mathrm{Tr}\big(g, \mathrm{Hom}_H(E, U)\big) = \frac{1}{|H|} \sum_{h \in H} a(h)\, \mathrm{Tr}\big((g, h^{-1}), U\big) \qquad (g \in G). \tag{5.3.1}$$

Proof. It suffices to treat an actual representation U. On $E^\vee \otimes_{\mathbf{Q}_\ell} U$, let H act diagonally and let G act through U. The averaging projector $|H|^{-1} \sum_h h$ on $E^\vee \otimes_{\mathbf{Q}_\ell} U$ has image $\mathrm{Hom}_H(E, U)$ and

commutes with g. The trace of g on $\mathrm{Hom}_H(E,U)$ is therefore the trace of the composite of g with the projector on $E^\vee \otimes_{\mathbf{Q}_\ell} U$, namely

$$\frac{1}{|H|}\sum_{h\in H} a(h^{-1})\,\mathrm{Tr}\big((g,h),U\big).$$

Replace h by h^{-1}. Exactness of invariants in characteristic zero extends the identity to virtual representations. □

Proposition 5.4. *There exists $V \in K_0(\mathbf{Q}_\ell[G])$ such that*

$$\mathrm{Tr}(g,V) = -\,\mathrm{length}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/I_g(X)_x) \qquad (g\neq 1).$$

Proof. Choose L/K, H, V_0 and V_1 as in Lemma 5.2. By Serre's theorem [20, §19.2, Theorem 44(i)], a_H is the character of a $\mathbf{Q}_\ell$-representation Art_H. Applying Lemma 5.3 to the weighted sum in (5.2.1) gives

$$[[\mathcal{O}_\Delta, \mathcal{O}_{\Gamma_g}]]_C = \mathrm{Tr}(g,V_0) - \mathrm{Tr}\big(g, \mathrm{Hom}_H(\mathrm{Art}_H, V_1)\big) \qquad (g\neq 1). \tag{5.4.1}$$

Since $\mathrm{Hom}_H(\mathrm{Art}_H,-)$ is exact over $\mathbf{Q}_\ell$, the class

$$V = -V_0 + \mathrm{Hom}_H(\mathrm{Art}_H, V_1) \in K_0(\mathbf{Q}_\ell[G])$$

is an integral virtual representation. Its trace is the negative of the intersection number in (5.4.1). Lemma 5.1 identifies that intersection number with the fixed-point length, proving the proposition. □

Positivity and conclusion

As in Serre's treatment of the classical case [20, §19.1], positivity follows from integrality and character orthogonality.

Lemma 5.5. *Suppose that $V \in K_0(\mathbf{Q}_\ell[G])$ satisfies $\mathrm{Tr}(g,V) = -i(g)$ for $g\neq 1$, where i is a nonnegative integer-valued central function on $G\setminus\{1\}$. Then*

$$W = V - \dim(V^G)\mathbf{Q}_\ell[G]$$

is the class of a representation, and its character is $-i(g)$ for $g\neq 1$, and $\sum_{g\neq 1} i(g)$ at 1.

Proof. The regular character vanishes off 1, and $\dim(W^G)=0$. Hence the inner product of W and the trivial character is 0, so the character of W at 1 is $\sum_{g\neq 1} i(g)$. Viewing character values in $\mathbf{C}$, the multiplicity of an irreducible character χ in W is

$$\frac{1}{|G|}\sum_{g\neq 1} i(g)\big(\chi(1) - \mathrm{Re}\,\chi(g)\big) \geq 0,$$

where we use $i(g)=i(g^{-1})\in\mathbb{Z}$ and $\mathrm{Re}\,\chi(g)\leq\chi(1)$. These multiplicities are integers since W is a virtual $\mathbf{Q}_\ell$-representation. Serre's Proposition 33 [20, §12.1] then shows that W is the class of an actual $\mathbf{Q}_\ell$-representation. □

Proof of Theorem 1.1. If the residue characteristic of $\mathcal{A}$ is zero, then it is easy to see that $i_{\mathcal{A}}(g) = 1$ for every $g \neq 1$, and $a_{\mathcal{A},G}$ is the augmentation character. If $\mathcal{A}$ has characteristic $p > 0$, apply Abe's theorem [2].

Suppose $\mathcal{A}$ has mixed characteristic. By Proposition 2.2, we may assume $\mathcal{A} = \mathcal{O}_{X,x}$ for a finite type affine G-scheme X over a dvr C, with uniformizer p, and $x \in X$ a closed point with residue field $k = C/pC$. Then by Proposition 4.3, we may assume that $\mathrm{Crit}(X/C)$ is supported at x. We may now apply Proposition 5.4 to (X, x) to obtain $V \in K_0(\mathbf{Q}_\ell[G])$ with

$$\mathrm{Tr}(g, V) = -i_{\mathcal{A}}(g) \qquad (g \neq 1).$$

The function $i_{\mathcal{A}}$ is central and nonnegative, and $I_g(\mathcal{A}) = I_{g^{-1}}(\mathcal{A})$ gives $i_{\mathcal{A}}(g) = i_{\mathcal{A}}(g^{-1})$. Lemma 5.5 therefore gives a representation with character (1.1), as required. □

References

[1] A. Abbes, *Cycles on arithmetic surfaces*, Compos. Math. **122** (2000), 23–111; published article.

[2] T. Abe, *On the Serre conjecture for Artin characters in the geometric case*, Invent. Math. **241** (2025), 605–622; arXiv:2405.19601.

[3] M. Artin, *Algebraic approximation of structures over complete local rings*, Publ. Math. Inst. Hautes Études Sci. **36** (1969), 23–58; published article.

[4] D. Beraldo and M. Pippi, *Proof of the Deligne–Milnor conjecture*, arXiv:2410.02327v1 (2024).

[5] D. Beraldo and M. Pippi, *Categorification of the localized intersection product and Bloch conductor formula*, arXiv:2511.10527v1 (2025).

[6] A. Blanc, M. Robalo, B. Toën and G. Vezzosi, *Motivic realizations of singularity categories and vanishing cycles*, J. Éc. polytech. Math. **5** (2018), 651–747; published article.

[7] S. Bloch, *Cycles on arithmetic schemes and Euler characteristics of curves*, in *Algebraic geometry, Bowdoin, 1985*, Proc. Sympos. Pure Math. **46**, Part 2, Amer. Math. Soc., 1987, 421–450; published article.

[8] P. Deligne and N. Katz, *Groupes de monodromie en géométrie algébrique, II* (SGA 7 II), Lecture Notes in Math. **340**, Springer, 1973; Exposé XIII by P. Deligne and Exposé XVII by N. M. Katz; published volume.

[9] P. M. Eakin, Jr., *The converse to a well known theorem on Noetherian rings*, Math. Ann. **177** (1968), 278–282; published article.

[10] R. Elkik, *Solutions d'équations à coefficients dans un anneau hensélien*, Ann. Sci. École Norm. Sup. (4) **6** (1973), 553–603; published article.

[11] A. Grothendieck et al., *Revêtements étales et groupe fondamental* (SGA 1), Lecture Notes in Math. **224**, Springer, 1971; revised edition.

[12] A. Hattori, *Rank element of a projective module*, Nagoya Math. J. **25** (1965), 113–120; published article.

[13] K. Kato, S. Saito and T. Saito, *Artin characters for algebraic surfaces*, Amer. J. Math. **110** (1988), 49–75; published article.

[14] K. Kato and T. Saito, *On the conductor formula of Bloch*, Publ. Math. Inst. Hautes Études Sci. **100** (2004), 5–151; published article.

[15] K. Kato and T. Saito, *Ramification theory for varieties over a local field*, Publ. Math. Inst. Hautes Études Sci. **117** (2013), 1–178; published article.

[16] M. Kisin, *Endomorphisms of logarithmic schemes*, Rend. Sem. Mat. Univ. Padova **109** (2003), 247–281; published article.

[17] Q. Lu and W. Zheng, *Categorical traces and a relative Lefschetz–Verdier formula*, Forum Math. Sigma **10** (2022), e10, 1–24; published article.

[18] B. Poonen, *The moduli space of commutative algebras of finite rank*, J. Eur. Math. Soc. **10** (2008), 817–836; author's version.

[19] J.-P. Serre, *Sur la rationalité des représentations d'Artin*, Ann. of Math. (2) **72** (1960), 405–420; published article.

[20] J.-P. Serre, *Linear representations of finite groups*, Graduate Texts in Math. **42**, Springer, 1977; published volume.

[21] M. Spivakovsky, *A new proof of D. Popescu's theorem on smoothing of ring homomorphisms*, J. Amer. Math. Soc. **12** (1999), 381–444; published article.

[22] *The Stacks Project*, §49.11 and Lemma 66.14.2.

[23] J. Tate, *Homology of Noetherian rings and local rings*, Illinois J. Math. **1** (1957), 14–27; published article.

[24] B. Toën and G. Vezzosi, *Trace and Künneth formulas for singularity categories and applications*, Compos. Math. **158** (2022), 483–528; arXiv:1710.05902.