Essential dimension of geometric local systems

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Abstract

Let $f : X \to B$ be a smooth projective family over a smooth complex variety, and let $\mathbb{H} = (R^m f_* \mathbb{Z})/\text{tors}$. We prove that there is an integer N such that, whenever $p^n \geq N$, the essential dimension at p of $\mathbb{H}/p^n$ is at least the dimension of its period image. This proves the Brosnan–Fakhruddin conjecture for these local systems. The proof compares the geometric Sen operator with the Higgs field. Integral Hodge–Tate theory gives a bound uniform in large primes. For each remaining prime, a bounded rational comparison at one geometric point suffices when n is large.

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1 Introduction

Let B be a smooth irreducible complex algebraic variety, and let $\mathbb{H}$ be an integral variation of Hodge structure on B. For a positive integer N, the finite local system $\mathbb{H}/N = \mathbb{H} \otimes \mathbb{Z}/N\mathbb{Z}$ determines a finite étale cover of B. Its essential dimension at a prime p is the least dimension of a variety from which this cover can be obtained by rational pullback, after a generically finite base change of degree prime to p. We recall the field-theoretic definition in §3.

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Motivated by the results for Shimura varieties in [6], Brosnan and Fakhruddin conjectured that, for a polarized integral variation with period map Φ, there is an integer N_0 such that

$$\mathrm{ed}_p(\mathbb{H}/p^n) \geq \dim \Phi(B) \qquad \text{if } p^n \geq N_0 \tag{1.1}$$

[3, §1.5, Conjecture 1]. We prove this assertion when $\mathbb{H}$ is the full cohomology of a smooth projective family, modulo torsion.

Theorem A. *Let $f : X \to B$ be a smooth projective morphism, with B a smooth irreducible complex variety, and fix $m \geq 0$. Put $\mathbb{H} = (R^m f_* \mathbb{Z})/\mathrm{tors}$, and let d be the dimension of its period image. There is an integer $N \geq 2$ such that, for every prime p and every integer $n \geq 1$ with $p^n \geq N$,*

$$\mathrm{ed}_p(\mathbb{H}/p^n) \geq d.$$

We first prove the inequality for $\mathbb{H}/p$ at sufficiently large primes. We then treat the finitely many remaining primes, allowing the required exponent to depend on p. We do not know how to extend the result to an arbitrary variation of Hodge structure and this remains an interesting question.

1.1 A sketch of the proof

For simplicity we sketch only the argument for p sufficiently large. The argument for the remaining primes is given in §3.3. It is in fact somewhat easier, since the result is only required to hold with n sufficiently large. A standard argument reduces us to a statement for a proper smooth family over a finite extension of $\mathbb{Q}_p$, however it will be useful to give a precise formulation. Let K be a finite extension of $\mathbb{Q}_p$, with ring of integers $\mathcal{O}_K$ and residue field k. Set $C = \widehat{\overline{K}}$. Let $f : \mathcal{X} \to \mathcal{B}$ be smooth projective, where $\mathcal{B}$ is smooth over $\mathcal{O}_K$ and its generic and special fibres are geometrically integral. Assume that the Hodge cohomology sheaves $R^a f_* \Omega^b_{\mathcal{X}/\mathcal{B}}$ are locally free and commute with base change. Assume also that the relative de Rham cohomology and its Hodge filtration are locally free and commute with base change, and that $\mathbb{T} = R^m f_{K,\text{ét}*} \mathbb{Z}_p$ is finite free.

Write

$$E^i = R^{m-i} f_* \Omega^i_{\mathcal{X}/\mathcal{B}}, \qquad \theta_i : E^i \longrightarrow E^{i-1} \otimes \Omega^1_{\mathcal{B}/\mathcal{O}_K}$$

for the Hodge bundles and their Higgs field, induced by the Gauss–Manin connection. We set $E^i = 0$ outside $0 \leq i \leq m$. The rank of the Higgs field means the rank of the contraction map

$$\theta : T_{\mathcal{B}/\mathcal{O}_K} \longrightarrow \bigoplus_i \mathrm{Hom}(E^i, E^{i-1}), \qquad D \longmapsto (\theta_i(D))_i. \tag{1.1.1}$$

Let d_0 be its generic rank on the special fibre.

Theorem B. *Under these hypotheses,*

$$\mathrm{ed}_{C,p}(\mathbb{T}/p|_{\mathcal{B}_C}) \geq d_0 \qquad \text{if } p > m + 3.$$

To prove Theorem B, let $\mathfrak{D} = \mathrm{Spf}\, \mathcal{O}_C\langle z \rangle$. We again write $f : X \to B$ for the generic fiber of f, and we denote by $\widehat{\mathcal{X}} \to \widehat{\mathcal{B}}$ the p-adic completion of f. Suppose that $s : B \to Z$ is a rational map over C such that $\dim Z < d_0$, and $\mathbb{T}/p$ descends to Z. (We ignore here the auxiliary prime to p covering, which is straightforward to deal with.) The idea is to construct a map of formal schemes $\gamma : \mathfrak{D} \to \widehat{\mathcal{B}}$ such that $s \circ \gamma|_C$ is well defined and constant. Then $\gamma^*(\mathbb{T}/p)$ is constant, and the theory

of the geometric Sen operator gives a contradiction, provided the pullback of the Kodaira–Spencer map θ is non-zero modulo $p^{\varepsilon''}$ at the origin, where $\varepsilon' \in \mathbb{Q}$ satisfies

$$\varepsilon := 1/(p-1) < \varepsilon' < 1-(m+1)\varepsilon = \varepsilon''.$$

We normalize $v(p)=1$ and fix a scalar $p^{\varepsilon'} \in C$ of valuation ε'. To construct γ we choose a suitable point $x \in \widehat{\mathcal{B}}(\mathcal{O}_C)$ and a derivation D of $\mathcal{O}_{\mathcal{B}}$ (defined on an open containing x) such that $D(\mathcal{O}_Z)=0$, and $x^*(\theta(D))$ is non-zero modulo the maximal ideal of $\mathcal{O}_C$, and define γ by

$$\gamma^*(a) = \sum_{j=0}^{\infty} \frac{p^{\varepsilon' j}}{j!} x^* D^j(a) z^j.$$

Crucially, the existence of γ satisfying these properties does not depend on the integral properties of the map s. Such a dependence would be fatal, as s depends on p.

Let us explain the use of the geometric Sen operator in more detail. Write $\mathbb{D}$ for the generic fiber of $\mathfrak{D}$. For a finite free $\mathbb{Z}_p$-local system $\mathbb{L}$ on $\mathbb{D}$, put $V^+ = \mathbb{L} \otimes \widehat{\mathcal{O}}^+_{\mathbb{D}}$ and $V = V^+[1/p]$, where $\widehat{\mathcal{O}}^+_{\mathbb{D}}$ is the completed integral structure sheaf on the pro-étale site of $\mathbb{D}$. Its geometric Sen operator is a map $\mathsf{S}_{\mathbb{L}} : V \to V \otimes \Omega^1_{\mathbb{D}/C}(-1)$, where (-1) denotes the Tate twist. If $\mathbb{L}/p$ is constant and $p>2$, Proposition 2.1.6 gives

$$\mathsf{S}_{\mathbb{L}}(V^+) \subset pV^+ \otimes_{\mathcal{O}_C} \mathcal{O}_C\, dz(-1). \tag{1.1.2}$$

Apply this to $\mathbb{L} = \gamma^*\mathbb{T}$ so $\mathbb{L}/p$ is constant. Under the hypotheses of Theorem B, we use Scholze's results [14] to control the comparison between the étale and Hodge lattices in V, and Howe's theorem [9] identifies the graded Sen operator with $\theta_\gamma = \gamma^*(\theta) \circ d\gamma$. These comparisons turn (1.1.2) into

$$v(p^{\varepsilon'}\gamma^*(\theta(D))|_0) = v\big(\theta_\gamma(\partial_z)|_0\big) \geq 1-(m+1)\varepsilon > \varepsilon' \tag{1.1.3}$$

(Proposition 2.4.1). This is a contradiction as $v(\gamma^*(\theta(D))|_0) = 0$.

In fact, in the proof, it is more convenient to take x to be the geometric point associated with the valuation at the generic point of the special fibre of $\mathcal{B}$. So x is defined over a complete algebraically closed extension C'/C. The disc $\mathfrak{D}$ is then defined over $\mathcal{O}_{C'}$, but the argument above remains unchanged.

We remark that it should be possible to adapt the above proof, at least for sufficiently large p, to other versions of integral p-adic Hodge theory, for example Faltings' integral crystalline theory [5], or the prismatic theory of Bhatt–Scholze [2]. As remarked above, the key observation is that the map γ above can be chosen without any assumptions on the integral behaviour of the map s. We leave this as an exercise for the reader.

1.2 Relation to earlier work

In the paper [6] with Farb and Wolfson, we used deformation theory at ordinary points to prove incompressibility of congruence covers. Fakhruddin and Saini extended the finite group scheme method to certain nonordinary cases in [4]. At the time of writing [6], it was already clear to us that, morally speaking, there should be an argument with the Kodaira–Spencer map, but we were unable to make this work without assuming good reduction properties of the map s.

The paper [10] proves p-incompressibility of principal congruence covers of Shimura varieties at all sufficiently large good primes. For exceptional groups the results of the present paper do not contain those of [10], as the corresponding variations of Hodge structures are not known to be motivic.

In [7], Farb, Kisin, and Wolfson use prismatic cohomology to obtain lower bounds for essential dimension from differential forms. Here we compare the Hodge bundles with the lattices induced by the local system on its Hodge–Tate graded pieces. Proposition 2.3.1 gives the required bound by retaining the integral Kummer complexes in Scholze's relative Hodge–Tate comparison [14]. We use the local integral calculation and décalage estimates of Bhatt, Morrow, and Scholze [1], in the relative form of Gaisin and Koshikawa [8]. Geometric Sen theory [13] and Howe's theorem [9] then relate this comparison to the Higgs field. We apply these results directly to the family over the formal disc.

The argument at a fixed prime uses only the rational comparison and geometric Sen theory. It is given in §3.3; §3.4 combines the two arguments to prove Theorem A.

AI use

Extensive use of GPT Astra was made both to arrive at the proofs of the results presented here, and to write a first draft of the manuscript. Significant human effort was needed to check and refine the arguments, and make the paper more readable. Part of this work was done at *The Overhang*, held in San Francisco on 9–10 September 2026. We thank the organizers and other participants for a very pleasant experience.

2 p-adic Hodge theory

2.1 Constancy and the geometric Sen operator

We first prove an integral estimate on a disc.

Let C be any complete algebraically closed extension of $\mathbb{Q}_p$. Write $\mathcal{O}_C$ for its valuation ring and $\mathfrak{m}_C$ for the maximal ideal, and normalize $v(p) = 1$. Fix compatible p-power roots of unity and set

$$\varepsilon = \frac{1}{p-1}.$$

For a rational number a, the notation $p^a M$ denotes the multiple of an $\mathcal{O}_C$-module M by a scalar of valuation a; it is independent of the scalar chosen. If M is finite free, and $m \in M$, we write $v(m)$ for the minimum of $v(\lambda(m))$ over linear maps $\lambda : M \to \mathcal{O}_C$ (which is well defined).

An almost zero module is a module killed by $\mathfrak{m}_C$. The almost category is the quotient by these modules; a superscript a denotes passage to this category. When working there we suppress this superscript. If M is finite free and $x \in M[1/p]$ satisfies $\mathfrak{m}_C x \subset p^a M$, then $v(x) \geq a$.

For a smooth rigid space B/C, write $B_{\text{pro\'et}}$ for its pro-étale site and $\widehat{\mathcal{O}}_B^+$ for the completed integral structure sheaf. Put $\widehat{\mathcal{O}}_B = \widehat{\mathcal{O}}_B^+[1/p]$. For a finite free $\mathbb{Z}_p$-local system $\mathbb{T}$ on B, put

$$V^+ = \mathbb{T} \otimes_{\mathbb{Z}_p} \widehat{\mathcal{O}}_B^+, \qquad V = V^+[1/p].$$

The geometric Sen operator is the $\widehat{\mathcal{O}}_B$-linear Higgs field

$$\mathsf{S}_{\mathbb{T}} : V \longrightarrow V \otimes_{\mathcal{O}_B} \Omega^1_{B/C}(-1) \tag{2.1.1}$$

of [13]. Here (i) is the usual Tate twist.

Let t be an indeterminate and set

$$A^+ = \mathcal{O}_C\langle t, t^{-1}\rangle, \quad A_\infty^+ = \mathcal{O}_C\langle t^{\pm 1/p^\infty}\rangle, \quad A = A^+[1/p], \quad A_\infty = A_\infty^+[1/p].$$

The second ring is the p-adic completion of the union of the finite-level rings. The map $Y = \mathrm{Spa}(A_\infty, A_\infty^+) \to \mathrm{Spa}(A, A^+)$ is an affinoid perfectoid pro-étale cover with group $\Gamma = \mathbb{Z}_p(1)$. Let $\sigma \in \Gamma$ be the generator satisfying $\sigma(t^{1/p^j}) = \zeta_{p^j} t^{1/p^j}$ for all j.

We also write v for the Gauss valuation on A_∞ and for the minimum valuation of the entries of a matrix over A_∞.

Lemma 2.1.2. *Let $\mathbb{T}$ be a finite free $\mathbb{Z}_p$-local system on $\mathrm{Spa}(A, A^+)$. Suppose $\mathbb{T}/p^n$ is constant. For every rational $0 \le \beta < n$, the module $H^0(Y, V^+)$ has an A_∞^+-basis in which the matrix of σ is congruent to 1 modulo p^β.*

Reference. Put $h = \mathrm{rk}\,\mathbb{T}$. A choice of basis of the constant local system $\mathbb{T}/p^n$ gives an isomorphism of sheaves

$$V^+/p^n \simeq (\widehat{\mathcal{O}}^+/p^n)^h.$$

By [16, Lemma 5.9], applied with $a = n$, the module $M = H^0(Y, V^+)$ is finite free of rank h and, for $0 < \beta < n$, the trivialization of V^+/p^n induces a Γ-equivariant isomorphism

$$M/p^\beta M \simeq (A_\infty^+/p^\beta)^h.$$

and the matrix of σ on $M/p^\beta M$ is 1 mod p^β in this basis. The proof in loc. cit. uses only almost acyclicity on affinoid perfectoid objects and almost algebra, so applies over the present field C. □

Lemma 2.1.3. *Let M be a finite free A_∞^+-module of rank h with continuous semilinear Γ-action. If its matrix P for σ satisfies $v(P-1) \ge \beta > \varepsilon$, an integral change of basis makes P a matrix $P_0 \in 1 + p^\beta M_h(A^+)$.*

Proof. Projection onto the monomials with integral exponents gives a bounded A-linear decomposition

$$A_\infty = A \oplus N. \tag{2.1.4}$$

The projection to A has norm one. On a monomial t^{u/p^s} with $s \ge 1$ and $p \nmid u$, the operator $\sigma - 1$ acts by multiplication by $\zeta_{p^s}^u - 1$. The operator $(\sigma-1)^{-1}$ on the C-linear span of these monomials is bounded, so extends to its completion N. Consequently

$$v\big((\sigma-1)^{-1}x\big) \ge v(x) - \varepsilon \qquad (x \in N). \tag{2.1.5}$$

Write $P = 1 + D + E$ using (2.1.4), with D and E in $M_h(A)$ and $M_h(N)$ respectively. Suppose $v(E) \ge s \ge \beta$. Put

$$Q = -(\sigma-1)^{-1}E, \qquad B = 1 + Q, \qquad P' = B^{-1}P\sigma(B).$$

By (2.1.5), $v(Q) \ge s - \varepsilon > 0$; both B and its inverse are integral. Using $\sigma(B) - B = -E$, we first compute

$$B(P'-1) = (P-1)\sigma(B) - E = D + (D+E)\sigma(Q).$$

Subtracting $BD = D + QD$ gives

$$B(P' - 1 - D) = (D+E)\sigma(Q) - QD.$$

Both terms on the right have valuation at least $s + \beta - \varepsilon$. Multiplying by the integral matrix B^{-1} gives the same bound for $P' - 1 - D$. Now replace P by P' and repeat the above. Each iteration increases $v(P-1-D)$ by at least $\beta - \varepsilon$. The products of the changes of basis converge in $\mathrm{GL}_h(A_\infty^+)$, and taking the limit of the matrices P gives the required P_0. □

Proposition 2.1.6. *Let $\mathbb{T}$ be a finite free $\mathbb{Z}_p$-local system on the generic fibre $\mathbb{D}$ of $\mathfrak{D} = \operatorname{Spf} \mathcal{O}_C\langle z\rangle$. If $\mathbb{T}/p^n$ is constant and $n > \varepsilon$, then*

$$\mathsf{S}_{\mathbb{T}}(V^+) \subset p^n V^+ \otimes_{\mathcal{O}_C} \mathcal{O}_C\, dz(-1)$$

as an inclusion of sheaves on $\mathbb{D}_{\text{proét}}$.

Proof. The formal opens where z, respectively $1+z$, is invertible cover $\mathfrak{D}$. Each is isomorphic to $\operatorname{Spf} A^+$, with $t = z$, respectively $t = 1+z$. On these opens $d\log t$ is a unit multiple of dz, so it suffices to prove the estimate on $\operatorname{Spa}(A, A^+)$.

Fix a rational β with $\varepsilon < \beta < n$. Lemmas 2.1.2 and 2.1.3 give an integral basis of $M = H^0(Y, V^+)$ whose A-span M_0 is Γ-stable and satisfies $M_0 \otimes_A A_\infty = M[1/p]$. Thus M_0 is a decompletion of V on the Kummer tower. The action of σ on M_0 is A-linear, with matrix $P_0 \in 1 + p^\beta M_h(A^+)$, where $h = \operatorname{rk}\mathbb{T}$. Let $\sigma^\vee \in \mathbb{Z}_p(-1)$ be dual to σ. By [13, Proposition 3.1.2 and proof of Proposition 3.2.3], the geometric Sen operator is obtained by extending

$$\begin{aligned} M_0 &\longrightarrow M_0 \otimes_A \Omega^1_{A/C}(-1), \\ m &\longmapsto (\log P_0)m \otimes d\log t \otimes \sigma^\vee \end{aligned}$$

$\widehat{\mathcal{O}}$-linearly. The jth term of the logarithm has valuation at least $j\beta - v_p(j) \geq \beta$, since $\beta > 1/(p-1)$. This proves that

$$\mathsf{S}_{\mathbb{T}}(V^+) \subset p^\beta V^+ \otimes_{\mathcal{O}_C} \mathcal{O}_C\, dz(-1).$$

The proposition now follows by intersecting the right hand side of this inclusion over all $\beta < n$, and noting that $\bigcap_{\delta>0} p^{-\delta}\widehat{\mathcal{O}}^+ = \widehat{\mathcal{O}}^+$. □

Lemma 2.1.7. *For a finite free $\mathbb{Z}_p$-local system on B, the geometric Sen operator commutes with extension of complete algebraically closed constant fields, and with pullback along a morphism of smooth rigid spaces over a field.*

Proof. Pullback over a fixed field is [13, Theorem 1.0.3(5)]. For an extension C'/C, work locally on a toric chart of B. The construction in [13, Proposition 3.2.3] computes the operator by differentiating the action of an open subgroup of the Kummer Galois group on a finite projective module at a finite level of the tower. Scalar extension to the infinite tower recovers V from this module. Completed scalar extension from C to C' preserves this description: the action matrices, their convergent logarithms, Kummer differentials, and Tate twists all commute with it. This proves the assertion. □

2.2 The Hodge–Tate filtration

We compare the lattices supplied by the local system and the Hodge bundles. The comparison will be applied to the family on the formal disc constructed in §3.

Let $K/\mathbb{Q}_p$ be finite, and let $f_K : X_K \to B_K$ be the analytification of a smooth projective morphism of smooth K-varieties. Fix $m \geq 0$, and put

$$\mathbb{T}_K = (R^m(f_K)_{\text{ét}*}\mathbb{Z}_p)/\text{tors}, \qquad E_K^i = R^{m-i}(f_K)_*\Omega^i_{X_K/B_K}.$$

The relative de Rham comparison [14, Theorem 8.8 and Remark 8.9] identifies the filtered bundle with connection associated to $\mathbb{T}_K[1/p]$ with $R^m(f_K)_*\Omega^\bullet_{X_K/B_K}$. Write $\theta_i : E_K^i \to E_K^{i-1} \otimes \Omega^1_{B_K/K}$ for its Higgs field.

Fix an embedding of K into C, and let $g : B \to (B_K)_C$ be a morphism of smooth rigid spaces over C. Write $g_K : B \to B_K$ for its composite with $(B_K)_C \to B_K$, and put

$$\mathbb{T} = g_K^* \mathbb{T}_K, \quad E_B^i = g_K^* E_K^i, \quad V^+ = \mathbb{T} \otimes_{\mathbb{Z}_p} \widehat{\mathcal{O}}_B^+, \quad V = V^+[1/p].$$

We also write θ_i for the pullback Higgs field on E_B^i, with values in $E_B^{i-1} \otimes \Omega^1_{B/C}$.

Let $\mathscr{C}_B$ be the pullback of $\mathrm{gr}^0 \mathcal{OB}_{\mathrm{dR},B_K}$ to $B_{\mathrm{pro\acute{e}t}}$ along g_K, with coefficients in $\widehat{\mathcal{O}}_B$. Taking the degree-zero graded part of the relative de Rham comparison and pulling back gives

$$\iota : V \otimes_{\widehat{\mathcal{O}}_B} \mathscr{C}_B \xrightarrow{\sim} \bigoplus_{j=0}^{m} E_B^j \otimes_{\mathcal{O}_B} \mathscr{C}_B(-j). \tag{2.2.1}$$

Define $F_i V$ to be the inverse image of the sum over $j \le i$ under $v \mapsto \iota(v \otimes 1)$. This is the Hodge–Tate filtration

$$0 = F_{-1}V \subset F_0 V \subset \cdots \subset F_m V = V. \tag{2.2.2}$$

Thus $F_i = \mathrm{Fil}_{\mathrm{HT}}^{-i}$ in decreasing notation. By [9, proof of Lemma 3.2.1], the graded maps of ι^{-1} are the scalar extensions to $\mathscr{C}_B$ of isomorphisms

$$\kappa_i : E_B^i \otimes \widehat{\mathcal{O}}_B(-i) \xrightarrow{\sim} \mathrm{gr}_i^F V.$$

Lemma 2.2.3 (Howe)**.** *The geometric Sen operator satisfies*

$$\mathsf{S}_{\mathbb{T}}(F_i V) \subset F_{i-1} V \otimes \Omega^1_{B/C}(-1).$$

Under κ_i and κ_{i-1}, its induced graded map is θ_i.

Proof. Over B_K this is [9, Lemma 3.2.1]. For B the assertion follows by pulling back along g_K, using Lemma 2.1.7. □

2.3 The integral bound

Put $X = X_K \times_{B_K} B$, and suppose that $f : X \to B$ is the rigid generic fibre of a smooth proper morphism $f : \mathfrak{X} \to \mathfrak{B}$ of smooth p-adic formal schemes locally of finite type over $\mathcal{O}_C$. Assume that all $R^a f_* \Omega^b_{\mathfrak{X}/\mathfrak{B}}$ are locally free and commute with base change, and that $R^m f_{\text{ét}*} \mathbb{Z}_p$ is torsion-free. It then identifies with $\mathbb{T}$. Put

$$E^i = R^{m-i} f_* \Omega^i_{\mathfrak{X}/\mathfrak{B}}, \qquad E_i^+ = E^i \otimes \widehat{\mathcal{O}}_B^+(-i), \qquad \lambda = \zeta_p - 1.$$

Here differentials on formal schemes are continuous, and $E^i \otimes \widehat{\mathcal{O}}_B^+$ denotes pullback along the specialization map $B_{\mathrm{pro\acute{e}t}} \to \mathfrak{B}_{\text{ét}}$. The generic fibre of E^i is E_B^i. We regard E_i^+ as a submodule of $\mathrm{gr}_i^F V$ via κ_i.

Proposition 2.3.1. *In the almost category there is an $\widehat{\mathcal{O}}_B^+$-submodule $\mathscr{H} \subset V^+$ such that*

$$\lambda^m V^+ \subset \mathscr{H} \subset V^+, \qquad \mathrm{gr}_i^F \mathscr{H} = \lambda^i E_i^+ \qquad (0 \le i \le m). \tag{2.3.2}$$

Here $F_i \mathscr{H} = \mathscr{H} \cap F_i V$, and the equality of graded pieces is taken inside $\mathrm{gr}_i^F V$.

Proof. We work in the almost category. To take relative pro-étale cohomology before proper push-forward, use the relative topos of [8, §2.1 and Proposition 5.7], with its maps

$$X_{\text{proét}} \xrightarrow{\nu_f} \mathscr{D} = B_{\text{proét}} \times_{\mathfrak{B}_{\text{ét}}} \mathfrak{X}_{\text{ét}} \xrightarrow{\pi} B_{\text{proét}}, \qquad q : \mathscr{D} \longrightarrow \mathfrak{X}_{\text{ét}}.$$

Here $\pi \circ \nu_f = f_{\text{proét}}$ and q is the other projection. We equip $\mathscr{D}$ with the sheaf of rings $\widehat{\mathcal{O}}^+_{\mathscr{D}} = \nu_{f*}\widehat{\mathcal{O}}^+_X$, and take pullbacks to $\mathscr{D}$ with coefficients in this ring. Put

$$K^+ = R\nu_{f*}\widehat{\mathcal{O}}^+_X, \qquad \widetilde{K}^+ = L\eta_\lambda K^+, \qquad \Omega^j_{\mathscr{D}} = q^*\Omega^j_{\mathfrak{X}/\mathfrak{B}}.$$

The décalage functor $L\eta_\lambda$ is represented, on a nonnegative complex $P^\bullet$ of λ-torsion-free modules, by

$$(\eta_\lambda P)^j = \{x \in \lambda^j P^j : dx \in \lambda^{j+1} P^{j+1}\}.$$

Since $H^0(K^+) = \nu_{f*}\widehat{\mathcal{O}}^+_X$ is λ-torsion-free, there is a natural map $\beta : \widetilde{K}^+ \to K^+$ [1, Lemma 6.10]. Put $\mathscr{H} = H^m(R\pi_*\widetilde{K}^+)$, and let $\mathscr{H}_i$ be the image of $H^m(R\pi_*\tau_{\le i}\widetilde{K}^+)$ in $\mathscr{H}$. The relative primitive comparison isomorphism [14, Theorem 1.3 and proof of Theorem 8.8(i)] identifies $H^m(R\pi_*K^+)$ with V^+. Thus β induces $b : \mathscr{H} \to V^+$, which is an isomorphism after inverting p. We will identify the graded pieces of $\mathscr{H}$ and bound its image under b above and below.

The relative integral Hodge–Tate comparison [8, Theorem 5.19] gives

$$H^j(\widetilde{K}^+) \simeq \lambda^j \Omega^j_{\mathscr{D}}(-j). \tag{2.3.3}$$

Now take a formally étale $\mathfrak{B}$-morphism $\mathfrak{U} \to \widehat{\mathbb{G}}^r_{m,\mathfrak{B}}$, where $\mathfrak{U}$ is an open of $\mathfrak{X}$, and let u be the pullback of one of the toric coordinates. Let $c(u)$ be the image in $(R^1\nu_{f*}\widehat{\mathcal{O}}^+_X)(1)$ of the Kummer class of u on the generic fibre of $\mathfrak{U}$. Using (2.3.3), one has

$$H^1(\beta)(1)(\lambda\, d\log u) = \lambda\, c(u)$$

[1, Definition 8.2], [8, (5.34)]. Taking exterior products then determines the map on cohomology induced by β in every degree. By (2.3.3) and coherent base change [8, Corollary 5.27], canonical truncation of $\widetilde{K}^+$ gives the spectral sequence

$$E_2^{a,j} = \lambda^j R^a f_* \Omega^j_{\mathfrak{X}/\mathfrak{B}} \otimes \widehat{\mathcal{O}}^+_B(-j) \implies H^{a+j}(R\pi_*\widetilde{K}^+). \tag{2.3.4}$$

We next compare the filtration on $\mathscr{H}[1/p]$ with F. Let $\mathscr{C}_X$ be the pullback of $\operatorname{gr}^0 \mathcal{OB}_{\text{dR},X_K}$ along $X \to X_K$, with coefficients in $\widehat{\mathcal{O}}_X$. The graded relative Poincaré resolution [14, Proposition 8.5] gives a quasi-isomorphism

$$f^*_{\text{proét}}\mathscr{C}_B \xrightarrow{\sim} \big(\mathscr{C}_X \xrightarrow{\bar\nabla_{X/B}} \mathscr{C}_X \otimes \Omega^1_{X/B}(-1) \longrightarrow \cdots\big). \tag{2.3.5}$$

Here $\bar\nabla_{X/B}$ is the $\widehat{\mathcal{O}}_X$-linear Higgs field induced from the connection on $\mathcal{OB}_{\text{dR},X_K}$ by taking the associated graded. The pullback on the left has coefficients in $\widehat{\mathcal{O}}_X$. The proof of [14, Lemma 8.7(iv)] shows that each term $\mathscr{C}_X \otimes \Omega^j_{X/B}(-j)$ is ν_{f*}-acyclic, with direct image $\Omega^j_{\mathscr{D}}[1/p] \otimes \pi^*\mathscr{C}_B(-j)$. The complex obtained from 2.3.5 by applying ν_{f*} has zero differentials: the coefficients in $\widehat{\mathcal{O}}^+_{\mathscr{D}}[1/p]$ and the pulled-back algebra $\pi^*\mathscr{C}_B$ are killed by $\bar\nabla_{X/B}$. Applying $R\nu_{f*}$ to the resolution therefore gives

$$K^+[1/p] \otimes \pi^*\mathscr{C}_B \simeq \bigoplus_{j\ge 0} \Omega^j_{\mathscr{D}}[1/p] \otimes \pi^*\mathscr{C}_B(-j)[-j], \tag{2.3.6}$$

where tensors are over $\widehat{\mathcal{O}}^+_{\mathscr{D}}[1/p]$. The algebra $\mathscr{C}_B$ is locally polynomial over $\widehat{\mathcal{O}}_B$, hence faithfully flat. Thus canonical truncation through degree i on the left of (2.3.6) corresponds to the sum with $j \leq i$ on the right.

Applying $R\pi_*$ to (2.3.6) and taking H^m gives (2.2.1). Indeed, coherent base change identifies the summands on the right, and the relative de Rham comparison is constructed from the same Poincaré resolution [14, proof of Theorem 8.8(ii)]. The definition of F and faithful flatness of $\mathscr{C}_B$ now give $b(\mathscr{H}_i[1/p]) = F_iV$.

The cohomology maps in (2.3.6) are compatible with (2.3.3) via the map β. Indeed, after twisting by (1), the inverse degree-one map sends $d\log u$ to $c(u)\otimes 1$, up to sign [14, Lemma 6.17]. This agrees with $H^1(\beta)(1)$ on $\lambda\Omega^1_{\mathscr{D}}$ after scalar extension to $\pi^*\mathscr{C}_B$. Exterior products give agreement in every degree. In particular $b[1/p]$ induces the maps κ_i, up to signs, on associated gradeds. Now (2.3.6) implies rational degeneration of (2.3.4) which then implies integral degeneration, since the E_2 terms are p-torsion-free. Thus $\mathscr{H}$ and $\mathscr{H}/\mathscr{H}_i$ are p-torsion-free. As $b[1/p]$ is an isomorphism, b embeds $\mathscr{H}$ in V^+; the equality $b(\mathscr{H}_i[1/p]) = F_iV$ then gives $\mathscr{H}_i = \mathscr{H} \cap F_iV$. Under this identification, the preceding comparison gives

$$\mathrm{gr}^F_i\,\mathscr{H} = \lambda^i E^+_i \subset \mathrm{gr}^F_i\, V.$$

Finally, by [1, Lemma 6.9], multiplication by λ^m on $\tau_{\leq m}K^+$ factors through $\tau_{\leq m}\widetilde{K}^+$. Apply $R\pi_*$ and take H^m; truncation above m does not affect this cohomology. It follows that $\lambda^m V^+ \subset \mathscr{H}$. □

2.4 The estimate on a disc

Keep the hypotheses of Theorem B, and let C'/C be a complete algebraically closed extension. Put $\mathfrak{D} = \mathrm{Spf}\,\mathcal{O}_{C'}\langle z\rangle$, and write $\mathbb{D}$ for its rigid generic fibre. For an $\mathcal{O}_K$-morphism $\gamma : \mathfrak{D} \to \widehat{\mathcal{B}}$, let $\gamma_\eta : \mathbb{D} \to \mathcal{B}^{\mathrm{an}}_K$ be its generic fibre. The pullback family $f_\gamma : \widehat{\mathcal{X}} \times_{\widehat{\mathcal{B}}} \mathfrak{D} \to \mathfrak{D}$ has Hodge bundles $E^i_\gamma = \gamma^* E^i$ and local system $\mathbb{T}_\gamma = \gamma^*_\eta \mathbb{T}$. Write

$$\theta_{\gamma,i} : E^i_\gamma \longrightarrow E^{i-1}_\gamma \otimes \Omega^1_{\mathfrak{D}/\mathcal{O}_{C'}}$$

for its Higgs field, and put $\theta_\gamma = \bigoplus_i \theta_{\gamma,i}$.

Proposition 2.4.1. *If $\mathbb{T}_\gamma/p^n$ is constant on $\mathbb{D}$ and $n > \varepsilon$, then*

$$v\big(\theta_\gamma(\partial_z)|_0\big) \geq n - (m+1)\varepsilon.$$

Proof. Put $V^+_\gamma = \mathbb{T}_\gamma \otimes \widehat{\mathcal{O}}^+_{\mathbb{D}}$. By Proposition 2.1.6, the contraction $S = \mathsf{S}_{\mathbb{T}_\gamma}(\partial_z)$ sends V^+_γ into $p^n V^+_\gamma(-1)$. Let $\mathscr{H}_\gamma \subset V^+_\gamma$ denote the lattice $\mathscr{H}$ of Proposition 2.3.1, applied to the pullback family f_γ over $\mathcal{O}_{C'}$. Then

$$p^{-n}\lambda^m S(\mathscr{H}_\gamma) \subset \lambda^m V^+_\gamma(-1) \subset \mathscr{H}_\gamma(-1).$$

By Lemma 2.2.3, this operator lowers the filtration on $\mathscr{H}_\gamma$. The factors λ^i and λ^{i-1} in the source and target graded pieces give, after cancelling the Tate twists, an almost integral map

$$p^{-n}\lambda^{m+1}\theta_{\gamma,i}(\partial_z) : E^i_\gamma \otimes \widehat{\mathcal{O}}^+_{\mathbb{D}} \longrightarrow E^{i-1}_\gamma \otimes \widehat{\mathcal{O}}^+_{\mathbb{D}} \qquad (1 \leq i \leq m).$$

Evaluation at $z = 0$ gives the asserted bound, since $v(\lambda) = \varepsilon$. Almost inclusions suffice for this valuation inequality, as observed in §2.1. □

3 Compressions and bounded derivations

We recall the field-theoretic definition of essential dimension at p. For a finite étale F-scheme Y, where F/C is a finitely generated field, define $\mathrm{ed}_{C,p}(Y)$ to be the least $\mathrm{trdeg}_C L$ for which there exist a finite extension F'/F of degree prime to p, a finitely generated subfield $C \subset L \subset F'$, and a finite étale L-scheme Y_L such that $Y_{F'} \simeq (Y_L)_{F'}$. For a finite local system on an integral variety, we apply this definition to its étale space at the generic point.

3.1 Valuations and bounded derivations

Let C be a complete algebraically closed extension of $\mathbb{Q}_p$, and let F/C be a finitely generated extension of transcendence degree b. Equip F with a valuation v extending that of C, normalized by $v(p) = 1$, with $v(F^\times) = v(C^\times) = \mathbb{Q}$. Write $\widehat{F}$ for its completion. Choose a basis $\partial_1, \dots, \partial_b$ of $\mathrm{Der}_C(F)$ such that

$$v(\partial_j(a)) \geq v(a) \qquad (a \in F).$$

These derivations extend continuously to $\widehat{F}$ and preserve $\mathcal{O}_{\widehat{F}}$. Put

$$\mathcal{D}^+ = \bigoplus_{j=1}^{b} \mathcal{O}_{\widehat{F}} \partial_j, \qquad \mathcal{D} = \mathcal{D}^+[1/p].$$

For a finite extension F'/F, write $\widehat{F}'$ for its completion at a chosen extension of the valuation.

Lemma 3.1.1. *If $\widehat{F}'/\widehat{F}$ is finite of degree prime to p, then $\mathcal{O}_{\widehat{F}'}/\mathcal{O}_{\widehat{F}}$ is finite étale. Each ∂_j extends uniquely to a derivation of $\widehat{F}'$ preserving $\mathcal{O}_{\widehat{F}'}$.*

Proof. The value group of $\widehat{F}$ is divisible, so the ramification index is one. By Ostrowski's lemma [11, Chapter 11, Lemma 11.17], the defect is a power of p, as is the inseparable residue degree. Since $[\widehat{F}' : \widehat{F}]$ is prime to p, the residue extension is separable of degree $[\widehat{F}' : \widehat{F}]$.

By [15, Tag 04GK], the residue extension lifts to a finite étale $\mathcal{O}_{\widehat{F}}$-algebra $\mathcal{O}'$. Henselian lifting embeds $\mathcal{O}'$ in $\mathcal{O}_{\widehat{F}'}$, and equality of degrees gives $\mathrm{Frac}\,\mathcal{O}' = \widehat{F}'$. Since $\mathcal{O}'$ is normal, $\mathcal{O}' = \mathcal{O}_{\widehat{F}'}$. The assertion about derivations follows from formal étaleness; see [15, Tag 00UP]. □

For a finite free $\mathcal{O}_{\widehat{F}}$-module Λ, let v_Λ denote the minimum of the coordinate valuations in any $\mathcal{O}_{\widehat{F}}$-basis. We use this notation on $\Lambda[1/p]$ and after any valued extension of $\widehat{F}$. Let $\theta : \mathcal{D}^+ \to \Lambda$ be an $\mathcal{O}_{\widehat{F}}$-linear map whose reduction has rank d_0. Fix F'/F and an extension of the valuation for which $[\widehat{F}' : \widehat{F}]$ is prime to p.

Lemma 3.1.2. *Let $C \subset L \subset \widehat{F}'$ be a finitely generated field with $\mathrm{trdeg}_C L = r < d_0$. There is $D \in \mathcal{D}^+ \otimes_{\mathcal{O}_{\widehat{F}}} \mathcal{O}_{\widehat{F}'}$ such that $D|_L = 0$ and $v_\Lambda(\theta(D)) = 0$.*

Proof. Extend scalars to $\mathcal{O}_{\widehat{F}'}$, retaining the notation $\mathcal{D}$, $\mathcal{D}^+$, Λ, and θ. Choose a transcendence basis $x_1, \dots, x_r$ of L/C. The equations $D(x_j) = 0$ impose at most r linear conditions on $\mathcal{D}$. In characteristic zero they also imply $D|_L = 0$. Their solution space W has dimension at least $b - r$.

The quotient $\mathcal{D}^+/(W \cap \mathcal{D}^+)$ is finite and torsion-free over the valuation ring $\mathcal{O}_{\widehat{F}'}$, hence free by [15, Tags 0539 and 00NZ]. Thus $W \cap \mathcal{D}^+$ is a direct summand of $\mathcal{D}^+$ of rank $\dim W$.

The reduction of $W \cap \mathcal{D}^+$ has dimension at least $b - r > b - d_0$. It cannot be contained in the kernel of the reduction of θ, which has dimension $b - d_0$. Choose a vector in this reduction outside the kernel, and lift it to $D \in W \cap \mathcal{D}^+$. This derivation has the required properties. □

3.2 The exponential disc

Lemma 3.2.1. *Let D be a C-linear derivation of $\widehat{F}'$ preserving $\mathcal{O}_{\widehat{F}'}$, and let $c \in C$ satisfy $v(c) > \varepsilon$. For an indeterminate z, the map*

$$a \longmapsto \exp(czD)(a) = \sum_{j \geq 0} \frac{c^j D^j(a)}{j!} z^j$$

is a C-algebra embedding $\widehat{F}' \hookrightarrow \widehat{F}'\langle z \rangle$. It maps $\mathcal{O}_{\widehat{F}'}$ into $\mathcal{O}_{\widehat{F}'}\langle z \rangle$, and it fixes every element annihilated by D.

Proof. Since $v(\widehat{F}'^{\times}) = v(C^{\times})$, dividing $a \neq 0$ by a constant of the same valuation shows that $v(D(a)) \geq v(a)$. This also holds for D^j. Hence

$$v\left(\frac{c^j D^j(a)}{j! a}\right) \geq j\big(v(c) - \varepsilon\big), \tag{3.2.2}$$

using $v_p(j!) \leq j/(p-1)$. The series converges in $\widehat{F}'\langle z \rangle$, and the Leibniz rule shows that $\exp(czD)$ is a C-algebra homomorphism. Evaluation at $z = 0$ is a left inverse to $\exp(czD)$, so $\exp(czD)$ is injective. By (3.2.2), $\exp(czD)$ maps $\mathcal{O}_{\widehat{F}'}$ into $\mathcal{O}_{\widehat{F}'}\langle z \rangle$. If $D(a) = 0$, all positive-degree terms in the defining series vanish, so $\exp(czD)(a) = a$. □

Proof of Theorem B. After replacing $\mathcal{B}$ by an affine open meeting the generic point of its special fibre, write $\mathcal{B} = \operatorname{Spec} A$ and choose étale coordinates $t_1, \ldots, t_b$ over $\mathcal{O}_K$, where $b = \dim \mathcal{B}_K$. We may also trivialize the Hodge bundles and arrange that the reduction of (1.1.1) has rank d_0 throughout this open.

Take $F = \kappa(\mathcal{B}_C)$ with the Gauss valuation defined by the generic point η of $\mathcal{B}_{\overline{k}}$. It extends the valuation of C, has value group $\mathbb{Q}$, and has residue field $\kappa(\mathcal{B}_{\overline{k}})$. The coordinate derivations $\partial_j = \partial/\partial t_j$ preserve $A \otimes_{\mathcal{O}_K} \mathcal{O}_C$ and its localization at η. For $a \in F$, normalizing its numerator and denominator by constants of C and applying the quotient rule gives $v(\partial_j(a)) \geq v(a)$. The reductions of the ∂_j form a basis of $T_{\mathcal{B}_{\overline{k}}/\overline{k},\eta}$. Put

$$\Lambda = \bigoplus_i \operatorname{Hom}_{\mathcal{O}_{\widehat{F}}}(E^i \otimes_A \mathcal{O}_{\widehat{F}}, E^{i-1} \otimes_A \mathcal{O}_{\widehat{F}}).$$

The Higgs contraction induces $\theta : \mathcal{D}^+ \to \Lambda$ whose reduction has rank d_0.

Suppose that $\mathbb{T}/p$ admits a compression of dimension $r < d_0$. Thus there is an extension F'/F of degree prime to p and a subfield $C \subset L \subset F'$, of transcendence degree r, over which the finite étale space of $\mathbb{T}/p$ descends to a finite étale L-scheme Y_L. The algebra $F' \otimes_F \widehat{F}$ is a product of finite extensions of $\widehat{F}$. Since the sum of their degrees is prime to p, at least one factor has degree prime to p. Choose the corresponding valuation on F'; its completion $\widehat{F}'$ is this factor.

Choose the bounded derivation D of Lemma 3.1.2. Since $p > m+3$, we may choose $c \in C$ with

$$\varepsilon < v(c) < 1 - (m+1)\varepsilon.$$

Let C' be a completed algebraic closure of $\widehat{F}'$ and put $\gamma^* = \exp(czD) : \widehat{F}' \to \widehat{F}'\langle z \rangle$. The restriction $\gamma^*|_A$ takes values in $\mathcal{O}_{C'}\langle z \rangle$, and hence gives a formal map

$$\gamma : \operatorname{Spf} \mathcal{O}_{C'}\langle z \rangle \longrightarrow \widehat{\mathcal{B}}_{\mathcal{O}_{C'}}.$$

Here $\widehat{\mathcal{B}}_{\mathcal{O}_{C'}}$ is the p-adic completion of $\mathcal{B} \times_{\mathcal{O}_K} \mathcal{O}_{C'}$, and $(d\gamma)|_0(\partial_z) = cD$. Write $\gamma_\eta : \mathbb{D} \to \mathcal{B}_{C'}$ for the generic fibre, where $\mathbb{D} = \mathrm{Spa}(C'\langle z\rangle, \mathcal{O}_{C'}\langle z\rangle)$.

By the definition of the compression, the étale space of $\gamma_\eta^*(\mathbb{T}/p)$ is identified with $Y_L \times_L \mathbb{D}$. Since $D|_L = 0$, the map $\gamma^*|_L : L \to C'\langle z\rangle$ factors through the algebraically closed field C', so this étale space is constant. Its module operations are constant as well, since $\mathbb{D}$ is connected.

Proposition 2.4.1 now gives

$$v\big(\theta_\gamma(\partial_z)|_0\big) \geq 1 - (m+1)\varepsilon.$$

But this matrix is $c\theta(D)$ and $\theta(D)$ has a unit entry. Its valuation is $v(c) < 1 - (m+1)\varepsilon$, a contradiction. □

3.3 The estimate for an arbitrary prime

We now remove the restriction on p and the hypothesis of good reduction by passing to sufficiently high p-power levels. The rational Hodge–Tate comparison at a fixed analytic point gives a valuation estimate with a constant independent of the level and the compression; increasing the exponent absorbs this constant.

Let $K/\mathbb{Q}_p$ be finite, put $C = \widehat{\overline{K}}$, and let $f : X \to B$ be smooth projective, with B/K smooth and geometrically integral. Put

$$\mathbb{T} = (R^m f_{\text{ét}*}\mathbb{Z}_p)/\mathrm{tors}, \qquad E_K^i = R^{m-i} f_* \Omega^i_{X/B}.$$

The Gauss–Manin connection on $R^m f_* \Omega^\bullet_{X/B}$ induces a Higgs contraction

$$\theta : T_{B/K} \longrightarrow \bigoplus_i \mathrm{Hom}(E_K^i, E_K^{i-1}).$$

Let d be its generic rank.

Theorem 3.3.1. *There is an integer $n_0 \geq 1$ such that*

$$\mathrm{ed}_{C,p}(\mathbb{T}/p^n|_{B_C}) \geq d \qquad (n \geq n_0).$$

Proof. We may assume $d > 0$. After a finite extension of K, choose a closed analytic polydisc $U_K \subset B^{\mathrm{an}}$ in the rank-d locus of θ, with coordinates $t_1, \dots, t_b$ obtained by rescaling algebraic étale coordinates, where $b = \dim B$. Put $U = (U_K)_C$ and $A = C\langle t_1, \dots, t_b\rangle$. Specialize §3.1 to $F = \kappa(B_C)$ with the Gauss valuation induced by A and $\partial_j = \partial/\partial t_j$. Since the subfield $C(t_1, \dots, t_b) \subset F$ is dense in $\mathrm{Frac}\, A$, the inclusion $A \hookrightarrow \widehat{F}$ defines the Gauss point $\eta : \mathrm{Spa}(\widehat{F}, \mathcal{O}_{\widehat{F}}) \to U$.

Fix a completed algebraic closure C' of $\widehat{F}$, and write $\bar\eta \in U_{C'}(C')$ for the extension of η to C'. Put $(E_K^i)_{\widehat{F}} = \eta^* E_K^i$. The Higgs contraction at η is

$$\theta : \mathcal{D} \longrightarrow G_{\widehat{F}} := \bigoplus_i \mathrm{Hom}_{\widehat{F}}((E_K^i)_{\widehat{F}}, (E_K^{i-1})_{\widehat{F}}).$$

The image $\theta(\mathcal{D}^+)$ is a finitely generated torsion-free module over the valuation ring $\mathcal{O}_{\widehat{F}}$, hence free of rank d by [15, Tags 0539 and 00NZ]. To preserve this rank on reduction, extend an $\mathcal{O}_{\widehat{F}}$-basis of $\theta(\mathcal{D}^+)$ to an $\widehat{F}$-basis of $G_{\widehat{F}}$, and let $\Lambda \subset G_{\widehat{F}}$ be its $\mathcal{O}_{\widehat{F}}$-span. Then $\theta(\mathcal{D}^+)$ is a direct summand of Λ; in particular, the reduction of $\theta : \mathcal{D}^+ \to \Lambda$ has rank d.

We now replace Proposition 2.4.1 by an estimate with a constant depending on $\bar\eta$ and Λ. Put

$$V = \mathbb{T}_{\bar\eta} \otimes_{\mathbb{Z}_p} C', \qquad V^+ = \mathbb{T}_{\bar\eta} \otimes_{\mathbb{Z}_p} \mathcal{O}_{C'}.$$

Pulling (2.2.2) back to $\bar\eta$ gives a filtration F on V. The comparison maps κ_i of §2.2 identify $\mathrm{gr}_i^F V$ with $(E_K^i)_{\bar\eta}(-i)$. Let $M \subset \mathrm{Hom}_{C'}(V, V(-1))$ consist of the maps u with $u(F_iV) \subset F_{i-1}V(-1)$ for every i. Using these identifications, taking associated graded and cancelling the Tate twists defines

$$\rho : M \longrightarrow G_{\widehat F} \otimes_{\widehat F} C', \qquad u \longmapsto (\mathrm{gr}_i\, u)_i.$$

Since ρ extends to a C'-linear map on $\mathrm{Hom}_{C'}(V, V(-1))$, there is an integer $a \geq 0$ such that

$$p^a \rho\big(M \cap \mathrm{Hom}_{\mathcal{O}_{C'}}(V^+, V^+(-1))\big) \subset \Lambda \otimes_{\mathcal{O}_{\widehat F}} \mathcal{O}_{C'}.$$

Put $\mathbb{D} = \mathrm{Spa}(C'\langle z\rangle, \mathcal{O}_{C'}\langle z\rangle)$. For a map $\gamma : \mathbb{D} \to U_{C'}$ with $\gamma(0) = \bar\eta$, write $\theta_\gamma = \gamma^*\theta \circ d\gamma$. Lemmas 2.2.3 and 2.1.7 give

$$\rho\big(\mathsf{S}_{\gamma^*\mathbb{T}}(\partial_z)|_0\big) = \theta_\gamma(\partial_z)|_0.$$

Proposition 2.1.6 therefore gives

$$v_\Lambda\big(\theta_\gamma(\partial_z)|_0\big) \geq n - a \tag{3.3.2}$$

whenever $\gamma^*(\mathbb{T}/p^n)$ is constant and $n > \varepsilon$. The constant a is independent of γ and of the compression.

Now a compression of dimension $r < d$ gives a finite extension F'/F of degree prime to p and a subfield $C \subset L \subset F'$ with $\mathrm{trdeg}_C L = r$. As in the proof of Theorem B, choose a completion $\widehat{F'}$ with $[\widehat{F'} : \widehat F]$ prime to p, and embed $\widehat{F'}$ in the fixed C'. Lemma 3.1.2, with $d_0 = d$, gives $D \in \mathcal{D}^+ \otimes_{\mathcal{O}_{\widehat F}} \mathcal{O}_{\widehat{F'}}$ such that $D|_L = 0$ and $v_\Lambda(\theta(D)) = 0$. For $n > a + \varepsilon$, choose $c \in C$ with $\varepsilon < v(c) < n - a$. By Lemma 3.2.1, the restriction of $\gamma^* = \exp(czD)$ to A defines $\gamma : \mathbb{D} \to U_{C'}$ with $\gamma(0) = \bar\eta$ and $(d\gamma)|_0(\partial_z) = cD$. Since $D|_L = 0$, the argument in the proof of Theorem B shows that $\gamma^*(\mathbb{T}/p^n)$ is constant. But (3.3.2) would give

$$n - a \leq v_\Lambda\big(\theta_\gamma(\partial_z)|_0\big) = v(c) < n - a.$$

This contradiction proves the assertion for any integer $n_0 > a + \varepsilon$. □

3.4 Spreading out

Proof of Theorem A. The Higgs contraction (1.1.1) has generic rank $d = \dim \Phi(B)$. After spreading out in characteristic zero, choose a specialization $f_E : X_E \to B_E$ over a number field $E \subset \mathbb{C}$ whose Higgs contraction still has generic rank d. By [12, Theorem 10.2], applied to the frame torsors of $\mathbb{H}/p^n$, and [6, Lemma 2.2.4], essential dimension at p cannot increase under this specialization. We may therefore replace f by $f_E \times_E \mathbb{C}$, retaining the notation f and $\mathbb{H}$.

For each prime p, choose a place $v \mid p$ of E and put $K = E_v$, $C = \widehat{\overline K}$, $f_K = f_E \times_E K$, and $\mathbb{T} = (R^m f_{K,\text{ét}*}\mathbb{Z}_p)/\mathrm{tors}$. By the étale–Betti comparison isomorphism and invariance of essential dimension under extension of algebraically closed base fields [6, Lemma 2.2.5 and §2.2.6],

$$\mathrm{ed}_p(\mathbb{H}/p^n) = \mathrm{ed}_{C,p}(\mathbb{T}/p^n), \qquad n \geq 1. \tag{3.4.1}$$

After shrinking B_E and inverting an integer $M \geq 1$, spread out f_E to a smooth projective morphism $\mathcal{X} \to \mathcal{B}$, where $\mathcal{B}$ is smooth over $\mathcal{O}_E[1/M]$ with geometrically integral fibres. We may arrange that the Hodge and de Rham cohomology sheaves of $\mathcal{X}/\mathcal{B}$ and the Hodge filtration are locally free and commute with base change [15, Tags 0FM0 and 0B8W], and that the Higgs contraction has rank d on every fibre. Enlarge M to exclude the primes dividing the torsion in $R^m f_*\mathbb{Z}$. For $p \nmid M$, the base change to $\mathcal{O}_K$ then satisfies the hypotheses of Theorem B, with special-fibre Higgs rank d. Together with (3.4.1), this gives

$$\mathrm{ed}_p(\mathbb{H}/p) \geq d \qquad (p > p_0)$$

for some $p_0 \geq 2$. By [6, Lemma 2.1.6], $\mathrm{ed}_p(\mathbb{H}/p^n) \geq \mathrm{ed}_p(\mathbb{H}/p)$, so the same lower bound holds for every $n \geq 1$ when $p > p_0$.

For each prime $p \leq p_0$, Theorem 3.3.1 and (3.4.1) give the same bound for $n \geq n_0(p)$, for some integer $n_0(p) \geq 1$. This proves the theorem. □

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